L C Karpinski

JUNIOR HIGH SCHOOL MATHEMATICS

BOOK III

GENERAL MATHEMATICS FOR THE NINTH YEAR

BY

JOHN C. STONE, A.M.

HEAD OF THE DEPARTMENT OF MATHEMATICS, STATE NORMAL SCHOOL, MONTCLAIR, NEW JERSEY, AUTHOR OF THE TEACHING OF ARITHMETIC, CO-AUTHOR OF THE SOUTHWORTH-STONE ARITHMETICS, AND THE STONE-MILLIS ARITHMETICS, ALGEBRAS, AND GEOMETRIES

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PREFACE

This series of mathematical textbooks marks a new type of mathematics as to aims, purposes, and material. Unbiased by tradition, the author seeks to give the mathematics necessary in order to interpret the quantitative phases of modern life, met by the average intelligent person outside of his specialized vocation. That is, all topics and processes have been selected for the primary purpose of developing the *power* to see and the *habit* of seeing those quantitative relationships and spatial magnitudes necessary to a better understanding and appreciation of present-day life.

To do this, the series makes use of concepts and processes usually classed as arithmetic, algebra, geometry, and trigonometry; but it uses only such a part of these subjects as is needed to interpret and appreciate the references to them found in general reading or met in present-day social activities. It is the modern needs of the student, then, that are kept constantly in mind in the selection of topics, and not the development of the subject along traditional lines.

Book III is a course in "general mathematics" for the *ninth year,* not only designed for the last year of the junior high school, but for the first year of the four-year high school course. And while the book completes the three-book series for junior high schools, it takes up in review the formula, equation, and intuitive geometry given in **Book II**, in order that it may be used independently of the rest of the series by those schools that wish to give a course in general mathematics instead of a formal course in "first-year algebra."

The book gives a simple, but adequate, treatment of the essential parts of elementary algebra, the practical parts of intuitive geom-

etry of lines, angles, triangles, and similar figures. The work of intuitive geometry leads naturally to indirect measurement by scale drawings, similar figures, and trigonometric ratios. Chapter XIV gives drill in expressing and interpreting relationships expressed by fractions and per cent. The closing chapter is a discussion of methods of interpreting statistics and of the three ways of measuring the central tendency of scattered data.

This book will develop powers to analyze relations of quantity and space, and lead to a habit of thinking needed in everyday social life, and at the same time give a splendid foundation for those who are to pursue the subject further.

The author wishes to acknowledge his indebtedness to Mr. William Betz, East High School, Rochester, New York, for many valuable suggestions, particularly as to the content of Chapter XV.

JOHN C. STONE.

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JUNIOR HIGH SCHOOL MATHEMATICS

BOOK III

CHAPTER I

LETTERS USED TO REPRESENT NUMBERS

YOU have already learned how to express a rule in mensuration by letters. In this book you will learn to use numbers expressed by letters in new ways and to perform all the operations with them that you performed with the numbers that you used in arithmetic.

When numbers are expressed by letters we are said to be using **literal notation.** In all our later work in mathematics we shall have to know how to use literal notation.

1. THE ADVANTAGE OF LITERAL NOTATION

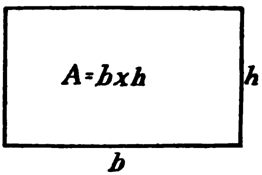
You learned in **Book II** some advantages of literal notation. Thus, you saw that by the use of letters you could express in a kind of *shorthand* any rule for areas or volumes. This *shorthand expression* was called a **formula** and you found that it saved a great deal of time when you wanted to express

a rule either orally or in writing. Thus, in finding the area of a rectangle, you learned in arithmetic that

The number of square units in the area of a rectangle is equal to the product of the number of linear units in its base and altitude.

You learned in **Books** I and II that this is more quickly and conveniently expressed by the formula

$$A = b \times h$$



You learned, too, in arithmetic, how to find the interest on any given principal for a given time at a given rate. Thus, you learned that

To find the interest on any given principal at a given rate for a given time, multiply the principal by the given rate and that product by the given time expressed in years.

Thus, the interest of \$900 at 6 % for 10 months is

$$\frac{10}{12} \times \frac{6}{100} \times \$900 \text{ or } \$45$$

This is expressed by a formula as

$$I = t \times r \times p$$

A letter used in a formula may represent any definite number of arithmetic and is thus called a **general number**. The following exercises give practice in making formulas.

1. Represent the cost C of n oranges at d cents each.

2. If a boy earned d dollars a week for n weeks, what will represent his total earnings for the given time?

3. If John has n marbles and buys s more, how many will he then have?

Since you cannot combine **n** *and* **s** *into a number represented by a single letter as you could 5 + 4, the sum is represented as* **n + s.**

4. If Frank had d dollars and earned t dollars more, how many dollars would he then have?

5. If a man's salary was s dollars, and was increased d dollars, what would it then be?

6. If from a plank f feet long t feet are cut off, how many feet remain?

7. A rectangle is b feet long and h feet wide. How many feet in its perimeter, or the distance around it?

8. If the side of a square is s feet, what is its perimeter?

9. At c cents per dozen, what will d dozen apples cost?

10. If d oranges cost c cents, what is the cost of each?

If 3 oranges cost 15 ¢, 1 orange will cost 15 ¢ ÷ 3 or 5 ¢. Likewise, if d oranges cost c cents, 1 orange will cost $c \div d$ cents.

When using literal numbers, the expression $c \div d$ is seldom used, but the fractional form of division is used instead. Thus, the expression for c divided by d would be written $\frac{c}{d}$. This is read "c over d."

11. When 3 apples cost d cents, what will 1 cost?

12. When n apples cost t cents, what will 1 cost?

13. When a man drives t hours at the rate of m miles per hour, how far does he drive?

14. If a man drives m miles in t hours, what is the average rate per hour?

15. When coal is s dollars per ton, how much will t tons cost?

16. When t tons of coal cost m dollars, what was the price per ton?

2. A NEW WAY TO REPRESENT MULTIPLICATION

In addition to the use of the sign $\times$, used in arithmetic, to indicate multiplication, a *dot* halfway up from the lower edge of the number-symbols is sometimes used between two literal numbers. Thus,

$$A = b \times h \text{ can be written } A = b \cdot h.$$

However, except when a possible confusion may arise, no sign whatever is used. Thus, $A = b \times h$ or $A = b \cdot h$ is usually written $A = bh$. And $2 \times w$ is usually written $2\,w$.

If an expression consists of two factors, either factor is called the **coefficient** (co-factor) of the other. In $3\,a$, 3 is the **numerical coefficient** of a. In ab, a is the **literal coefficient** of b, and b is likewise the literal coefficient of a.

The product of a, b, and c is written abc.

When the factors are alike, as $a \times a \times a$, the product is written a^3.

The 3 is called the **exponent** and denotes the number of times the **base** a is to be taken as a factor. Thus, when $a = 4$, $a^3 = 4 \times 4 \times 4 = 64$.

a^2 is read "a square" and means $a \times a$.

a^3 is read "a cubed" and means $a \times a \times a$.

a^4 is read "a fourth power" and means $a \times a \times a \times a$.

a is sometimes called "the first power of a"; a^2, "the second power of a"; a^3, "the third power of a"; and so on.

1. Give the value of x^2 when $x = 5$.
2. Give the value of y^3 when $y = 3$.
3. Give the value of b^4 when $b = 10$.
4. Give the value of $3\,t$ when $t = 15$.
5. Give the value of rs when $r = 6$ and $s = 7$.

6. What is the value of mnl when $m = 3$, $n = 4$, and $l = 10$?

7. Express the product of m and n in three ways. Tell which way you prefer and why.

8. When C is the total cost of l lemons at c cents each, express C in terms of l and c. Find C when $l = 9$ and $c = 5$.

9. Express by a formula the total distance D traveled in h hours at m miles per hour. In the formula find D when $h = 8$ and $m = 20$.

10. Express by a formula the total earnings E in n months at s dollars per month. Find E when $n = 8$ and $s = 250$.

11. Express by formula the interest I on p dollars at r % for t years. Find I when $p = 1000$, $r = 6$, and $t = \frac{3}{4}$.

12. At r dollars per week, how much can a boy earn in t weeks? Calling E the total earnings, write the formula. Find E when $r = 9$ and $t = 8$.

13. Find the value V of n bushels of apples at r dollars per bushel. Find V when $n = 20$ and $r = 2.25$.

14. If a boy rides at an average rate of 8 miles per hour, how far can he ride in 4 hours? In t hours? At an average rate of r miles per hour, how far can he ride in t hours?

15. Using D, r, and t for *distance*, *rate*, and *time*, express by formula the relations of problem 14.

16. Find the average rate of a man who is driving when he makes 125 miles in 5 hours. In t hours. When he drives d miles in t hours.

17. Using the notation of problem 15, write the formula that expresses the relations of problem 16.

18. An automobile traveling at the rate of 20 miles per hour will require what time to make a town 50 miles away? d miles away? At an average rate of r miles per hour, how long will it take to make a town d miles away?

19. Using the notation of problem 15, write the formula that expresses the relations of problem 18.

3. LITERAL EXPRESSIONS OF MORE THAN ONE TERM

In using literal notation in mathematics you will often need expressions of more than one term connected by plus or minus signs. Thus, the perimeter of a triangle whose sides are a, b, and c, respectively, is $a + b + c$. Or, if m is the maturity value of a note and d is the bank discount, then the proceeds are $m - d$.

Literal expressions of one term are **monomial** expressions; of two terms connected by plus or minus signs are **binomials**; and of three terms, **trinomials**. The general term **polynomial** is often used for all expressions of more than one term connected by plus or minus signs.

1. What is the perimeter of a rectangle whose dimensions are s and t? Since there are two sides each s, and two sides each t, the perimeter equals what binomial expression?

2. At simple interest, the amount due at any given time is the face of the note plus the interest due. Calling the amount due A, the face of the note f, and the interest due i, write A in terms of f and i.

3. James worked n hours in the morning and m hours in the evening. Write a binomial that expresses the total number of hours that he worked.

4. After spending c dollars, how much had one left from t dollars?

5. A boy having d dollars, earned t more and spent n; express as a trinomial the amount left.

Write expressions for the following:

6. The sum of twice a and three times b.

7. Half the difference between x and twice y.

8. The square of a less the cube of b.

9. The product of $3a$ and b.

10. One-half the sum of $2a$ and $5b$.

11. The sum of the squares of a and b.

12. The square of the sum of a and b.

NOTE. — To show that the sum as one number is to be squared, $a + b$ is inclosed in **parentheses**, and "the square of the sum" is written $(a + b)^2$.

13. The difference of the squares of x and b.

14. The square of the difference of a and b.

4. CONSTRUCTING MATHEMATICAL SYMBOLS

As you have already seen, formulas are used to express mathematical facts in brief form. These literal expressions not only aid the memory by picturing the fact more clearly to the eye, but they save much time in writing or expressing the facts. By literal expressions, too, the analysis and solution of a problem is often made more direct and simple.

To write a formula from a rule, or to state a relationship in terms of letters, two things are essential:

1. *Study the wording carefully to see the mathematical processes that are to be performed, and the order in which they are to be performed.*

2. *Then express by letters and mathematical symbols the facts that were expressed in words.*

Write a formula for the following rules of mensuration:

1. The number of square units in the area of a rectangle is the product of the number of linear units in its two dimensions. (A, b, h.)

NOTE. — The letters A, b, and h suggest the letters to be used in the formula. Thus let A = area, b = base, h = height.

2. The number of square units in the area of a triangle is half the product of the number of linear units in its base and altitude. (A, b, h.)

3. The number of square units in the area of a parallelogram is the product of the number of linear units in the base and altitude. (A, b, h.)

4. The number of square units in the area of a trapezoid is half the product of the number of linear units in the height multiplied by the sum of the linear units in the two bases. (A, b_1, b_2, h.)

5. The number of square units in the area of a square is the square of the number of linear units in a side of the square. (A, s.)

6. The circumference of a circle is π times its diameter. (C, d.)

7. The circumference of a circle is 2π times its radius. (C, r.)

8. The number of square units in the area of a circle is π times the square of the number of linear units in the radius. (A, r.)

9. The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two sides. (H, A, B.)

10. The volume of a cube is the cube of one of its edges.

NOTE. — This is but a brief and inaccurate statement of the fact that the number of cubic units in the volume of a cube is the cube of the number of linear units in one of its edges. Complete statements were made in problems 1–5, and 8–9. In the following, make the complete statement indicated by the brief statement.

11. The volume of a rectangular prism is equal to the product of its length, breadth, and height. (V, l, b, h.)

12. The volume of a rectangular prism is equal to the product of the area of the base and the height. (V, b, h.) (Make a complete statement of this rule.)

13. The volume of a regular pyramid is one-third of the product of the area of its base and height. (V, b, h.)

14. The volume of a right circular cylinder is π times the square of the radius times its height. (V, r, h.)

15. The volume of a right circular cylinder is the area of the base times its height. (V, b, h.)

16. The volume of a right circular cone is one-third of the product of the area of the base by the height. (V, b, h.)

17. The area of the curved surface of a cylinder is π times its diameter times its height. (S, d, h.)

18. The area of the curved surface of a cone is half the product of π times the diameter times the slant height of the cone. (S, d, s.)

19. The area of the surface of a sphere is four times π times the square of its radius. (S, r.)

20. The volume of a sphere is $\frac{4}{3}$ times π times the cube of its radius. (V, r.)

5. FORMULAS OF SCIENCE AND INDUSTRY

In the various sciences and industries are many formulas that the user *accepts as true* because they have been made by some one who is an authority. The one making a practical use of them in the science or industry is not concerned with how they are made, but takes them as accepted facts and uses them. The following illustrate miscellaneous formulas used in practical life.

1. The formula $C = \frac{E}{R}$ is much used in work with electricity. Compute C when $E = 20.5$ and $R = 16.75$.

2. The velocity of the recoil of guns is computed by the formula $V = \frac{wv}{W}$, where V = velocity of recoil, W = weight of gun and carriage, in pounds, w = weight of projectile, and v = muzzle velocity of projectile. A 10-inch gun on a battleship fires a 400-pound projectile with a muzzle velocity of 1600 feet per second. The weight of gun and carriage is 22 tons. Find the velocity of the recoil.

3. The force of pressure P of the wind, in pounds per square foot, is computed from $P = .005\,V^2$, where V = velocity of wind in miles per hour. Find the force of the wind when blowing at 40 miles per hour. What would be the total pressure of this wind against the side of a house 20 feet high and 60 feet long?

4. If an object, such as a brick dislodged from the wall, starts from rest and falls towards the earth, the distance that it will fall in a given length of time is computed by the formula $s = \frac{1}{2}at^2$, where s = distance in feet, $a = 32$, and t = number of seconds elapsed. Find the distance an object

will fall in 1 second; 2 seconds; 3 seconds; 4 seconds; 10 seconds; 60 seconds; 5 minutes.

NOTE. — This formula holds accurately only for bodies falling in a perfect vacuum. For bodies falling through the air, the velocity is somewhat diminished by the resistance of the air.

5. To measure temperature, two different kinds of thermometers are in use: the Fahrenheit and the Centigrade. On the former the freezing point is marked 32° and the boiling point 212°. On the latter these points are marked 0° and 100°, respectively. If the temperature is read on a Fahrenheit thermometer, the corresponding temperature on the Centigrade thermometer is computed by the formula $C = \frac{5}{9}(F - 32)$, where C = temperature in degrees on Centigrade scale and F = temperature in degrees on Fahrenheit scale.

When it is 70° by the Fahrenheit thermometer, what is the temperature on the Centigrade thermometer? When 64°? When 48°? When 80°?

6. The strength or capacity for work of engines is expressed in horse power. The horse power of steam engines is found by the formula $H.P. = \frac{plan}{33000}$, where p = pressure of steam in pounds per square inch, l = length of stroke in feet, a = area of piston in square inches, and n = twice the number of revolutions per minute. Compute the horse power of an engine in which a test shows $p = 95$ pounds, $l = 30$ inches, $a = 706.8$ square inches, and $n = 100$.

7. Find the horse power of a steam engine in which $p = 110$ pounds, $l = 24$ inches, $n = 120$, and the diameter of the piston is 16 inches.

SUGGESTION. — From the given diameter of the piston a must be computed. That is, you must find the area of a 16-inch circle.

8. The horse power of automobile engines is computed by the formula $H.P. = KND(D-1)(R+2)$, where $K = .197$ for commercial touring cars, N = number of cylinders, D = diameter of cylinders, and R = ratio of the stroke to the diameter. What is the horse power of a 4-cylinder engine of a touring car in which the diameter is 4 inches and the stroke 5 inches?

9. The number of ways that a committee of 3 persons may be selected from a group of n persons is $\frac{n(n-1)(n-2)}{6}$. In how many ways may a committee of 3 be selected from 4 persons? From 5 persons? From 6 persons?

10. The horse power that may be transmitted safely by a certain kind of shafting without breaking or twisting is computed by the formula $H.P. = \frac{nd^3}{64}$, where n = number of revolutions per minute and d = diameter of shafting in inches. How many horse power can be transmitted by such a shafting of 4-inch diameter making 76 revolutions per minute? By one of 5-inch diameter making 100 revolutions per minute?

11. When a brick arch is supported by a tie rod to keep the walls from spreading, the "horizontal thrust," or stretching force exerted on the rod, in pounds per linear foot of arch, is obtained by the formula $F = \frac{1.5\,WS^2}{R}$, where W = weight on arch in pounds per square foot, S = span of arch in feet, R = rise of arch in inches. Find the strain on the tie rod in an arch on which the weight is 360 pounds per square foot, the span 4 feet, and the rise of arch 18 inches.

12. The discharge of a pump in gallons per minute is obtained from the formula $G = .03264\ Td^2$, where $G =$ number of gallons, $T =$ travel (total distance traveled) of piston in feet per minute, $d =$ diameter of cylinder in inches. Suppose that the diameter of the cylinder of a pump is 18 inches, that the stroke of the piston is 24 inches, and that it makes 40 strokes per minute. Find the discharge.

6. HOW LITERAL NOTATION AIDS IN THE SOLUTION OF A PROBLEM

In many of the indirect problems of arithmetic, writing the relation by using some letter for the unknown quantity, or the number wanted, is a great aid in the analysis and solution of the problem. The following illustrations show a few advantages of using such notation. Such statements involving an unknown quantity are called **equations.**

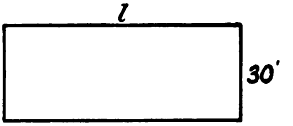
1. How long a rectangular plot 30 ft. wide will have the same area as a square plot whose side is 45 ft.?

SOLUTION

Let $l =$ the unknown length.

Then $30\,l =$ the area of the rectangle.

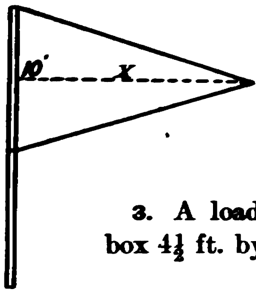
And since $2025 =$ the number of square feet in the square plot, we have the relation (equation),



$$30\,l = 2025.$$

Now we see that we have the product of two factors of which one factor is known, and we are to find the other. From the meaning of division, we know that $l = 2025 \div 30 = 67.5$.

2. A boy is making a triangular sail for his boat. It is to contain but 65 square feet. If he makes it 10 feet wide at the base, how long can it be from the base to the opposite corner?



SOLUTION

Let $x =$ the required length.

Then $\frac{10x}{2}$ or $5x =$ the area.

Hence $5x = 65$,

and $x = 13$.

3. A load of sand (1 cu. yd.) will fill a sand box $4\frac{1}{2}$ ft. by 8 ft. to what depth?

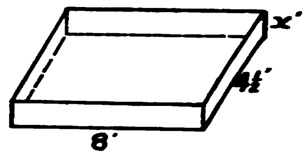
SOLUTION

Let $x =$ the required depth.

Then $36x =$ volume in cu. ft.

Hence $36x = 27$,

and $x = \frac{27}{36} = \frac{3}{4}$ (ft.) or 9 inches.



In the same way state and solve the following:

4. Using a string as the radius, some boys want to lay off a circular running track that will be $\frac{1}{8}$ of a mile (660 ft.) around. How long a string is needed?

5. A boy in a corn-raising contest wishes to see how much he can raise on just one acre (160 sq. rd.). The field is 20 rd. long. How wide a strip must he take?

6. Two boys are going to see who can produce more carrots from plots of equal size. One has a plot of 50 ft. by 90 ft. If the other boy uses a plot 120 ft. long, how wide a strip must he take to have the same area?

7. A boy has two boxes in which he keeps feed for his hens. One box is 14 in. wide and 18 in. long, and the other is 12 in. wide and 30 in. long. To what depth must he fill them to have just 4 bu. in each? (Use 1 bu. = 2150 cu. in.)

8. A boy wants a sail for his canoe that will contain 36 sq. ft. He can buy canvas 3 ft. wide. Not allowing for waste in making, how many yards must he buy?

9. A man wished to know to what depth his winter's supply of coal of 14 T. would fill his bin, which was 8 ft. by 12 ft. Allowing 35 cu. ft. to the ton, find it.

10. A man wished to cut off 5 lb. from a bar of lead 2 in. square. Allowing .4 lb. per cubic inch, find what length he must take.

11. Allowing 50 cu. ft. to the ton, to what depth will 40 T. of hay fill a rectangular mow 15 ft. by 18 ft.?

12. To what depth will 800 bu. of oats fill a bin 10 ft. by 20 ft.? (Allow 1.25 cu. ft. per bushel.)

13. A triangular plot with a base of 10 rd. contains $\frac{1}{2}$ of an acre. How far from the opposite corner to the base?

CHAPTER II

HOW TO SOLVE AND USE AN EQUATION

1. THE FORMAL SOLUTION OF AN EQUATION

You saw in **Book II** that an **equation** is a statement that two expressions have the same value. In the last chapter you solved indirect problems in mensuration by the use of the equation. In solving them you reasoned from the meaning of division how to find the value of the unknown factor.

In **Book II** you found four kinds of equations, the solutions of which required **subtraction, division, addition,** or **multiplication.** The four kinds are reviewed here.

1. Solve $n + 4 = 9$.

This equation states that "some number increased by 4 is equal to 9." To solve it is to find what the number is. Evidently it is 5, for $5 + 4 = 9$. While this is easily found by inspection, in more complicated problems we make use of the axiom that

If equals are subtracted from equals, the remainders are equal.

Hence a more formal solution is:

Given $\qquad n + 4 = 9.$

Subtracting 4 from each side of the equation,

$$n = 5.$$

2. Solve $3n = 15$.

This is a statement that "3 times some number is equal to 15." To solve it is to find what the number is. Evidently it is 5, for $3 \times 5 = 15$. But again, a more formal solution is based upon the axiom that

If equal numbers are divided by equal numbers (not zero), the quotients are equal.

Hence the solution is:

Given $$3n = 15.$$

Dividing both sides by 3,

$$n = 5.$$

3. Solve $x - 5 = 8$.

This is a statement that "some number less 5 is equal to 8." To solve it is to find what the number is. Evidently it is 13, for $13 - 5 = 8$. But this is solved more formally by the axiom that

If equals are added to equals, the sums are equal.

Given $$x - 5 = 8.$$

Adding 5 to both sides,

$$x = 13;$$

for $x - 5$ is 5 less than x, hence if 5 is added to $x - 5$ we have x.

4. Solve $\frac{x}{2} = 8$.

This is a statement that "some number divided by 2 is equal to 8." Evidently it is 16, for $16 \div 2 = 8$. More formally this is solved by the axiom that

If equals are multiplied by equals, the products are equal.

Given $$\frac{x}{2} = 8.$$

Multiplying both sides by 2,

$$x = 16.$$

To check the solution of an equation, substitute in the original equation the value found for the unknown and see if it makes both members equal.

5. Solve and check. $\frac{3x}{2} = 18.$

Given $$\frac{3x}{2} = 18.$$

Multiplying both members by 2,

$$3x = 36.$$

Dividing both members by 3,

$$x = 12.$$

CHECK

$$\frac{3 \times 12}{2} = 18.$$

$$3 \times 6 = 18.$$

$$18 = 18.$$

Solve and check:

1. $x + 7 = 13.$
2. $x - 22 = 40.$
3. $s + 15 = 26.$
4. $d + 10 = 25.$
5. $s + 7 = 16.$
6. $x - 5 = 10.$
7. $2x = x + 7.$
8. $6x = 5x + 3.$
9. $6 + x = 13.$
10. $9x = 6 + 8x.$
11. $11y - 17 = 10y.$
12. $7t = 28 + 6t.$
13. $x + 7 = 14.$
14. $x - 5 = 2.$

15. $a - 13 = 5.$

16. $y - 7 = 12.$

17. $c - 17 = 3.$

18. $x - 45 = 10.$

19. $y - 13 = 20.$

20. $t - 6 = 11.$

21. $s - 4 = 18.$

22. $r - 10 = 15.$

23. $\frac{y}{3} = 6.$

24. $\frac{x}{7} = 5.$

25. $\frac{s}{4} = 7.$

26. $\frac{n}{5} = 20.$

27. $\frac{a}{7} = 10.$

28. $\frac{c}{15} = 6.$

29. $\frac{r}{8} = 11.$

30. $\frac{x}{12} = 6.$

31. $\frac{x}{15} = 8.$

32. $\frac{x}{9} = 15.$

33. $\frac{s}{12} = 8.$

34. $\frac{r}{13} = 5.$

35. $\frac{x}{9} = 16.$

36. $\frac{r}{20} = 7.$

37. $\frac{d}{8} = 16.$

38. $\frac{t}{5} = 120.$

39. $\frac{x}{7} = 25.$

40. $\frac{x}{9} = 80.$

41. $\frac{n}{7} = 19.$

42. $\frac{r}{8} = 19.$

43. $\frac{s}{9} = 13.$

44. $\frac{r}{11} = 85.$

45. $3x = 24.$

46. $7y = 35.$

47. $2x = 26.$

48. $12t = 60.$

49. $5y = 45.$

50. $7t = 84.$

51. $11x = 132.$

52. $12x = 180.$

53. $8a = 96$.

54. $10b = 90$.

55. $7n = 105$.

56. $6x = 144$.

57. $5y = 175$.

58. $7s = 147$.

59. $3r = 195$.

60. $7x = 854$.

61. $3t = 672$.

62. $9x = 288$.

63. $5n = 305$.

64. $7x = 924$.

65. $8r = 368$.

66. $11n = 176$.

2. EQUATIONS REQUIRING MORE THAN ONE METHOD OF SOLUTION

The value of the unknown number of any equation is found by using one or more of the four methods you have studied. This will be shown in the following solutions:

1. Solve $5x + 12x + 112 = 10x + 196$.

Adding similar terms, $5x$ and $12x$,

$$17x + 112 = 10x + 196 \quad (1)$$

Subtracting 112 and $10x$ from both members,

$$7x = 84 \quad (2)$$

Dividing both members by 7, $x = 12$. (3)

Notice that by subtracting 112 from both members, we get all terms free of x out of the first member, and by subtracting $10x$ from both members, we get all terms which contain x out of the second member.

CHECK. — When $x = 12$, the first member becomes $5 \times 12 + 12 \times 12 + 112$, or 316, and the second member becomes $10 \times 12 + 196$, or 316, as it should.

2. Solve $13a - 14 + a = 34 + 8a$.

By combining similar terms, what does the first member become?

What must be added to get all terms free of a out of the first member of the result?

What must be subtracted to get the second member free of terms containing a?

What is your resulting equation?

By what must you then divide both members?

Evaluate both members by using the value of a obtained, and see if your result is correct.

3. Solve $\frac{n}{4} + \frac{n}{5} + 3 = \frac{n}{10} + 10$.

What is the least number that will contain each denominator?

By what then must each member be multiplied to free each term of fractions?

Show that the resulting equation is $5n + 4n + 60 = 2n + 200$.

Combining similar terms, what does this become?

What is necessary in order to get 60 out of the first member?

How is $2n$ gotten out of the second member?

Show that $n = 20$.

It is evident from the foregoing examples that the steps in the process of solving such an equation are as follows:

1. *If the equation contains one or more fractions, multiply both members by the least number that will contain all the denominators in order to free the equation of fractions.*

2. *Unite similar terms by adding, or subtracting, as the case may require.*

3. *Free the first member of all terms that do not contain the unknown number, and free the second member of all terms that do contain the unknown number, by adding the same number to both members or by subtracting the same number from both members. To remove a term that is added, subtract it, and to remove a term that is subtracted, add it to both members.*

4. *Divide both members of the resulting equation by the coefficient of the unknown number.*

5. *Always check the work by evaluating both members of the original equation, using the value found for the unknown number.*

Solve and check:

1. $3x + 4 = 2x + 7.$
2. $6a - 5 = 4a + 1.$
3. $5y + 5 + 3y = 8.$
4. $7b + 3 = b + 21.$
5. $3b + 4 = 25.$
6. $2x - 5 = 7 - x.$
7. $5c - 4 = 12 - 3c.$
8. $3y - 7 = 14 - 4y.$
9. $5A = 3A + 6.$
10. $20P - 5 = 5P + 25.$
11. $5h + 15 = h + 35.$
12. $4b - 3 = b.$
13. $5x - 5 = x + 3.$
14. $13x + 7 - 15 = 5x + 32.$
15. $12c - 13 + c = 35 + 7c.$
16. $13a - 7 + 16 = 5a + 18 - a.$
17. $\frac{2x}{3} + \frac{3x}{4} + 9 = \frac{5x}{6} + 16.$
18. $\frac{a}{2} + \frac{4a}{5} + 6 = \frac{a}{5} + 17.$
19. $\frac{3n}{4} - 7 = \frac{2n}{3} + 8.$
20. $2b + \frac{5b}{6} - 17 = 0.$
21. $c + \frac{c}{2} + \frac{c}{3} = \frac{c}{6} + 30.$
22. $\frac{2t}{3} - 5 + \frac{t}{4} = 7 + \frac{t}{12} + \frac{t}{6}.$
23. $\frac{a-6}{4} + \frac{a-4}{3} = \frac{17}{3}.$

24. $\dfrac{2n-2}{3}+\dfrac{n}{2}=\dfrac{5n+20}{6}+4.$ **26.** $\dfrac{2x}{3}+12=\dfrac{4x}{5}+6.$

25. $x-\dfrac{x}{2}=30+\dfrac{x}{6}.$ **27.** $\dfrac{x}{3}+\dfrac{3x}{5}-8=150-\dfrac{4x}{7}.$

28. $\dfrac{5x+2}{3}-\dfrac{3}{4}=\dfrac{x+5}{2}+\dfrac{13}{4}.$

3. PROBLEMS SOLVED BY EQUATIONS

You have already had some experience in solving problems by use of the equation. It is only the indirect problems of fractions, percentage, and mensuration that you had in arithmetic that were more easily solved by use of the equation. But many problems will arise later whose solutions without the use of algebra are very difficult.

The problems of the following exercises do not meet a real need in everyday life but are given for exercise in expressing a relationship in terms of an equation, in order to fit you for more important work that follows. You will find them interesting, however, for you will see how algebra enables you to solve problems that are almost impossible without it. The four problems show in detail how to solve problems by use of the equation.

1. A horse and wagon cost $375. The horse cost twice as much as the wagon. What was the cost of each?

There are two unknown numbers, the cost of the horse and the cost of the wagon.

Let $x =$ cost of wagon.

Then $2x =$ cost of horse,

because "the horse cost twice as much as the wagon."

Hence $2x + x = 375,$

because "the horse and wagon cost \$375."

Adding similar terms, $3x = 375.$

Dividing by 3, $x = 125.$

Multiplying by 2, $2x = 250.$

Hence the wagon cost \$125, and the horse cost \$250.

2. Three men contributed to an enterprise \$4800. The second contributed twice as much as the first, and the third contributed as much as the other two. How much did each contribute?

Let $a =$ amount contributed by the first.

Then $2a =$ amount contributed by the second. Why?

And $3a =$ amount contributed by the third. Why?

Hence $a + 2a + 3a = 4800.$ Why?

$6a = 4800.$ Why?

$a = 800.$ Why?

$2a = 1600.$ Why?

$3a = 2400.$ Why?

How much did each contribute?

Observe that we took the following steps in getting an equation:

1. *We first read the problem to discover what numbers were required.*

2. *We then let some letter represent one of the required numbers.*

3. *We then found statements in the problem that gave the other unknown numbers in terms of this letter.*

4. *We then found another statement in the problem that gave an equation.*

5. *This statement was expressed in the symbols of algebra as an equation.*

3. A house and lot cost \$8400. The house cost 3 times as much as the lot. Find the cost of each.

How many numbers are unknown? Which unknown number will you represent by a letter? What statement will give the other unknown number?

What statement in the problem gives an equation?

What, then, is the equation?

Solve the equation.

Study the following solution. Show how the steps outlined above were followed to get the equation. Then show how each new equation in the solution was obtained.

4. One-fourth of a certain number diminished by 9 is the same as 40 diminished by one-third of the number. Find the number.

Let n represent the number.

Then $\frac{n}{4}$ represents one-fourth of the number and $\frac{n}{3}$ represents one-third of it.

Then $\frac{n}{4} - 9 = 40 - \frac{n}{3}$. Why?

And $3n - 108 = 480 - 4n$. Why?

Hence $7n = 588$. Why?

And $n = 84$, the number. Why?

To state relationships, you must first be able to answer such questions as the following:

5. If x denotes a certain number, what will denote a number 5 larger? What will denote one 5 less? One $\frac{1}{5}$ as large?

6. My orchard yielded 100 barrels of apples more this year than last. If n represents last year's yield, what will represent this year's yield? If y represents the yield of this year, what will represent the yield of last year?

7. Henry's bicycle cost \$ y and John's cost three times as much. What represents the cost of John's?

8. A house cost \$ x and the lot cost one-third as much. What will represent the cost of the lot? What will represent the cost of both?

9. One boy sells twice as many papers as another. By the use of some letter represent the number sold by each. By both.

10. One horse cost \$10 more than another. By the use of some letter represent the cost of each. The cost of both.

11. John solved 3 more problems than James. Represent the number solved by each. By both.

12. If A has \$ x, B one-half as much, and C one-third as much, what will represent the amount all three have?

13. If a boy has 6 marbles and buys 5 more, how many has he? If he has a marbles and buys b more, how many has he?

14. One part of 16 is 12. What is the other part? One part of 16 is x. What is the other part? One part of a is n. What is the other part?

15. The difference between two numbers is 9, and the smaller is 7. What is the larger? The difference between two numbers is d, and the smaller is x. What is the larger?

16. How old will a man be in 12 years if his present age is 42? How old will he be in 12 years if his present age is x? How old will he be in y years if his present age is x?

17. A man's present age is 32. What was it 8 years ago? What was it x years ago? A man's present age is a. What was it b years ago?

18. One number is x and another is three times as great. What expresses the other number? One number is x and another is y times as great. What expresses the other number?

In each of the above examples you have expressed one number in terms of one or more other numbers.

Write as equations and solve:

19. \$5 added to my money will give me \$28. How much have I?

20. A watch and chain together cost \$125. The chain cost \$30. Find the cost of the watch.

21. Seven times a number less 5 equals 51. Find the number.

22. Eight times my money and \$40 more is 12 times my money. How much have I?

23. I lost $\frac{3}{8}$ of my money and had \$35 left. What had I at first?

24. I sold a house for \$6000. This was a gain of $\frac{1}{4}$ of the cost. Find the cost.

25. After traveling $\frac{5}{12}$ of the journey, I had 21 miles yet to go. What was the length of the whole journey?

26. I bought 6 lb. of coffee and 5 lb. of tea for \$6.40. The tea cost twice as much per pound as the coffee. Find the cost of each.

27. In a certain class there are 24 pupils, and there are twice as many girls as boys. How many boys are there?

28. Two boys sold 45 papers. One sold $\frac{4}{5}$ as many as the other. How many did each boy sell?

29. What number must be added to 32 in order that twice the sum shall be 106?

30. A rod 40 inches long is cut into two pieces. The shorter piece is $\frac{2}{3}$ as long as the other piece. Find the lengths of the pieces.

31. A certain number diminished by 40 is the same as 40 diminished by $\frac{1}{3}$ of the number. Find the number.

32. Divide 165 into two parts, one of which is 45 less than the other.

33. The sum of two numbers is 99, and their difference is 57. What are the numbers?

34. Three times a father's age is 8 times that of his son, and the sum of their ages 132. Find their ages.

SUGGESTION. — Let $3x$ = the son's age.

35. A rectangular field is twice as long as it is wide, and the distance around it is 672 yards. Find the width and length.

36. Find two numbers whose difference is 64, one of which is three times the other.

37. I bought 3 horses and 2 cows for $960. I paid twice as much for a horse as for a cow. What did I pay for each?

38. Three candidates at an election received 3600 votes among them. A received $\frac{4}{5}$ as many as B, and C received $\frac{3}{5}$ as many as B. How many did each candidate receive?

39. A number, its half, its third, and its fourth make 100. Find the number.

40. A farmer sold $\frac{1}{4}$ of his farm at \$60 per acre, $\frac{1}{8}$ of it at \$75 per acre, and $\frac{5}{8}$ of it at \$50 per acre. He received in all \$22,250. How many acres in the farm?

41. Divide 60 into two such parts that the greater shall equal 4 times the lesser.

42. Henry has 24 marbles and Paul has 14. How many must Henry give Paul in order that they may each have the same number?

43. A, B, and C buy a piece of property for \$3100. A invests twice as much as B, and C invests \$100 more than A and B together. How much does each person invest?

44. Two men were employed to dig a ditch 630 feet long. One dug an average of 45 feet a day, and the other an average of 60 feet a day. How long did it take them to dig the ditch?

45. Two trains start at the same time from stations 450 miles apart, one running at the rate of 35 miles per hour, the other at the rate of 40 miles per hour. In how many hours do they meet?

46. Find the number whose double exceeds 12 by as much as 9 exceeds the number.

47. A boy has \$1.05 in dimes and nickels. He has as many dimes as nickels. How many has he of each?

48. A merchant's profits doubled each year for three years. His profits for all three years were \$9450. What were his profits the first year?

49. A has \$35 more than B. B has $\frac{6}{11}$ as much as A. How much has each?

50. A makes a journey of 112 miles. He drives 5 times as far as he walks, and goes by train twice as far as he drives. How far does he go by train?

51. The sum of three angles of a triangle is 180°. If two of the angles of the triangle are equal, and the other is $\frac{2}{3}$ of their sum, how many degrees in each angle?

52. The difference between 4 times a certain number and $\frac{1}{4}$ of the number is 30. Find the number.

53. Two boys together solved 48 problems. One solved $\frac{7}{9}$ as many as the other. How many did each boy solve?

54. If to a certain number you add its half, its fourth, and 52 more, the sum is 5 times the original number. Find the number.

55. One side of a parallelogram is 16 inches, and the whole perimeter is 80 inches. Find the length of the adjacent side.

4. THE TRANSFORMATION OF FORMULAS

In the formula $A = bh$ for the area of a rectangle, A is called the **subject** of the formula just as in the rule for the area, the word "area" is the subject of the sentence. In using a formula to find some unknown value, not the subject, it is sometimes more convenient to transform the formula by solving for this letter in terms of the others; that is, to change the subject. Thus, in the formula

$$A = bh.$$

Dividing both sides by b,

$$\frac{A}{b} = h.$$

Or $$h = \frac{A}{b}$$

h now becomes the subject.

This new formula translated into a rule is,

To find the width of a rectangle whose area and base are known, divide the area by the base.

Transform the following formulas as required:

1. $A = bh$. Solve for b.
2. $A = \frac{1}{2} bh$. Solve for h.
3. $C = \pi d$. Solve for d.
4. $C = 2\pi r$. Solve for r.
5. $i = prt$. Solve for r.
6. $V = bwh$. Solve for h.
7. $V = \pi r^2 h$. Solve for h.
8. $V = \frac{1}{3} bh$. Solve for h.
9. $V = \frac{1}{3} \pi r^2 h$. Solve for h.
10. $A = \frac{1}{2}(b_1 + b_2)h$. Solve for h.

*11. $A = S^2$. Solve for S.

12. $A = \pi r^2 h$. Solve for r.
13. $V = \pi r^2 h$. Solve for r.
14. $V = \frac{1}{3} \pi r^2 h$. Solve for r.
15. $S = 4\pi r^2$. Solve for r.
16. $V = \frac{4}{3} \pi r^3$. Solve for r.
17. $h^2 = a^2 + b^2$. Solve for a.
18. $C = a^2 - b^2$. Solve for a.

NOTE. — You learned in chapter IX, **Book II**, that if you knew the square of a number, you could find the number by extracting the square root. Thus, if the square of some number is 49, the number is $\sqrt{49}$, or 7. Hence if $A = S^2$, then $S^2 = A$, and $S = \sqrt{A}$. Likewise, if $X^3 = Y$, $X = \sqrt[3]{Y}$.

CHAPTER III

LINES AND ANGLES

In every walk of life we have to consider the **measurement** of **lines, angles, surfaces,** and **solids.** A study of these requires a knowledge of that branch of mathematics **called geometry.** In this chapter we shall study the **measurement** of lines and angles and certain properties of them.

1. KINDS OF LINES

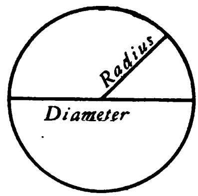
A line has **length** only, and not width or thickness. To represent a line to the eye, however, some kind of mark is necessary. To lay off a straight line we may lay a ruler down on a piece of paper and mark along the edge with a sharpened pencil. We call this a line, but in reality it is but the physical representation of the line. It merely locates it. We are concerned merely with its location or its length, not the width of the mark.

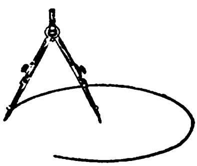
When two lines intersect, their intersection determines a **point** which has neither length, breadth, nor thickness, but merely **location** or **position.**

STRAIGHT LINE BROKEN LINE CURVED LINE

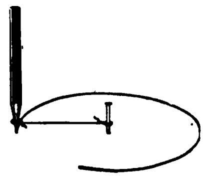
A **broken line** is made up of a number of parts each of which is straight. A **curved line** is a line no part of which is straight.

A **circle** is a closed curved line all points of which are equally distant from a point within called the **center.** The distances from all points of the circle to the center are called the **radii,** all of which are equal. Two radii forming a straight line are a **diameter.**





A **straightedge** or **ruler** is used for drawing straight lines, and **compasses** are used for drawing circles. A good substitute for compasses consists of a cord with a loop in one end in which a pencil or crayon may be inserted.



For the work of this chapter you need a straightedge and a pair of compasses.

2. THE STRAIGHT LINE

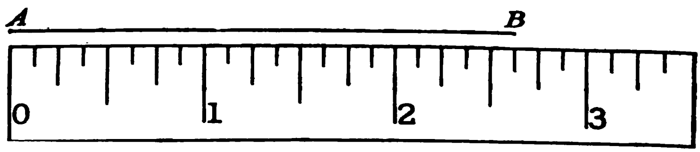
While the mark that represents a straight line is of necessity limited in length, a straight line must be thought of as extending indefinitely in either direction. That part of a straight line whose length we wish to determine is called a **line segment,** or, more briefly, a **segment.** Thus,

When speaking of the line AB we mean the straight line through points A and B with no thought of length. To refer to the part between A and B, we speak of the "segment AB." Thus, a segment has a definite beginning point and a definite ending point.

When a line has a beginning point but extends away from it indefinitely it is called a **ray.**

3. THE MEASUREMENT OF A LINE SEGMENT

To measure any magnitude is to apply some **unit of measure** to it and find how many times it contains the unit of measure. Thus, to measure segment AB, we apply the zero mark of our ruler to point A, and with the edge of the ruler along AB, read the length opposite the point B, which is $2\frac{5}{8}$ inches. Thus, the segment AB is $2\frac{5}{8}$ times as long as *1 inch*, the *unit of measure*.



To measure longer length, the **foot** or the **yard** is used as the unit of measure. The foot rules, yardsticks, and tape measures used are divided into fractional parts of the units used.

The inch, foot, and yard are the units of length used for most industrial and commercial purposes in this country, while in the sciences we use a system called the **metric system**. The metric system is the system used for both scientific and industrial purposes in most countries, except in this country and Great Britain.

4. THE METRIC SYSTEM OF MEASURES

Nearly all nations have derived their standard units of length from the lengths of some parts of the human body or from other familiar objects, as a grain of barley, the length of a pole used in fencing their farms, or the length of a step in marching, etc.

During the French Revolution the French Government

appointed a commission to devise a more scientific system, and one that would eliminate the inconveniences of the existing systems. The **metric system** is the result.

The **standard unit** of the metric system, from which all other units are derived, is the **meter**. An attempt was made to get an indestructible unit by surveying a long distance on the meridian through Paris and then computing the distance from the equator to the pole and taking one ten-millionth of this as the most convenient length for the **primary unit**. This resulted in a unit 39.3707 inches long for the meter. While later measurements show a slight error in fixing the length of the meter, it does not affect the usefulness of the system.

The system is a **decimal system**, for the number of units of one denomination which make one unit of the next higher denomination is always 10 or a power of 10. The relations of all larger or smaller units to the primary unit are shown by their names made from Greek and Latin prefixes. The Latin prefixes denote *parts* of the primary unit, and the Greek prefixes denote *multiples* of the primary unit.

LATIN PREFIXES

Milli meams $\frac{1}{1000}$; thus, millimeter means $\frac{1}{1000}$ of a meter.
Centi means $\frac{1}{100}$; thus, centimeter means $\frac{1}{100}$ of a meter.
Deci means $\frac{1}{10}$; thus, decimeter means $\frac{1}{10}$ of a meter.

GREEK PREFIXES

Deka mean 10; thus, dekameter means 10 meters.

Hekto means 100; thus, hektometer means 100 meters.

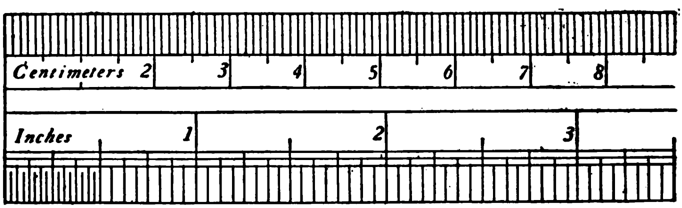
Kilo means 1000; thus, kilometer means 1000 meters.

Myria means 10,000; thus, myriameter means 10,000 meters.

TABLE OF LINEAR MEASURE

10 **millimeters** (mm.)	= 1 **centimeter** (cm.)
	= .3937079 inch
10 **centimeters**	= 1 decimeter (dm.)
10 decimeters	= 1 **meter** (m.)
	= 39.37079 inches.
10 **meters**	= 1 dekameter (Dm.)
10 dekameters	= 1 hektometer (Hm.)
10 hektometers	= 1 **kilometer** (Km.)
	= 3280.9 feet, or .62137 mile
10 **kilometers**	= 1 myriameter (Mm.)

NOTE. — The units in common use are in **bold-face type**. The abbreviations of the Latin prefixes begin with small letters and those of the Greek prefixes with capitals.



It is seen, then, that the great advantages of this system are: (1) *A decimal relation between the units so that all reductions may be made by annexing zeros or moving the decimal point;* and (2) *self-defining names of the derived units that save memorizing tables of relation as in our system.*

5. MEASUREMENTS ARE BUT APPROXIMATIONS

You have noticed in measurement that a high degree of accuracy is difficult. Some of the sources of error are: (1) The extremes of the segment measured may not fall exactly upon the divisions of the measure used and the dif-

ferences must be estimated; (2) the graduated scale used may not be exact; (3) the edge of the measure may not lie exactly along the line measured; (4) a tape line may stretch or contract; (5) the measuring scale may not be held rigidly and is thus allowed to slip. Great care will decrease the errors, but not wholly eliminate them.

1. Draw a straight line segment on the blackboard and let each student measure it. How nearly did your results agree?

2. Measure the length of a page in this book. How nearly do the lengths found by the class agree?

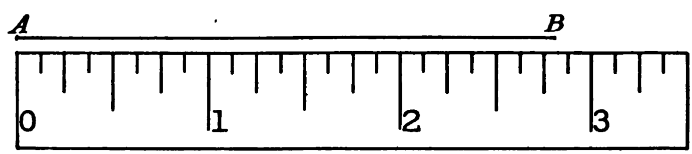
3. Measure the length of your desk and compare results.

4. Measure the width of the blackboard and compare results.

5. Using different yardsticks or rulers, measure the length of your schoolroom. How nearly do results agree?

6. If you are using a foot rule graduated to $\frac{1}{8}$ in., how would you estimate the length of a line segment, one end of which fell between two scales of the ruler?

7. Estimate as nearly as you can the length of AB.



6. THE PRODUCTS OF APPROXIMATE NUMBERS

As you have just seen, numbers derived from measurement are but approximate and hence when using them in finding areas or volumes the results obtained are no more dependable than the numbers used. This is shown in the problems that follow.

1. Find the area of a room whose dimensions are 18.6 ft. by 24.2 ft.

$$\begin{array}{r} 18.6 \\ \underline{24.2} \\ 37\;2 \\ 744 \\ \underline{372} \\ 450.12 \end{array}$$

(No. of sq. ft.)

Discussion. — 18.6 ft. means, in general, that the *exact* length lies somewhere between 18.55 and 18.64, and 6 is but an approximate number. In 24.2 the last 2 is also but approximate. Hence the first partial product (372) is approximate. In the same way it is seen that at least the right-hand figures of the second and third partial products are in doubt. Hence the zero of the quotient is in doubt and the *real* area is not known nearer than 450 with the last two figures in doubt.

In general, the product found for two approximate numbers is not dependable to the same order or denomination as the factors. Thus, in the problem given, the *tenths* of the factors were in doubt, while in the product the *ones'* digit was in doubt.

2. Find the circumference of a circle whose diameter is 12.8 ft.

$$\begin{array}{r} 3.1416 \\ \underline{12.8} \\ \mathit{251328} \\ 6283\mathit{2} \\ \underline{31416} \\ 40.\mathit{21248} \end{array}$$

Discussion. — In general, the .8 of 12.8 is in doubt, and as you know, the 6 of 3.1416 is but the *nearest* number of ten thousandths. Hence all the figures in italics are in doubt. Moreover, if they *were* all dependable, no instrument would measure .00008 ft., and if it could be measured, such a degree of accuracy would not be needed in practical work. It seems useless, then, to find them, and a way to save work will now be shown.

7. ABRIDGED PROCESS OF MULTIPLICATION

As you have just seen in the last two examples, most of the work was useless. Not more than 40 or 40.2 of the product of 40.21248 could be depended upon. Hence most of the work could have been saved. To do this we must multiply by the highest order first.

WORK

$$\begin{array}{l} 12.8 \\ \;\;3.1416 \\ \hline 38.4 \\ \;\;1.28 \\ \;\;\;.51 \\ \;\;\;.01 \\ \hline 40.2 \end{array}$$

Explanation. — To have *tenths* in the final product, the partial products were carried to *hundredths* so that the sum would be more accurate to tenths. Multiplying by 3, the first partial product is 38.4.

$.1 \times 12.8 = 1.28$; $.04 \times 12.8 = .51^+$

$.001 \times 12.8 = .01^+$, and $.0006 \times 12.8$ is not needed.

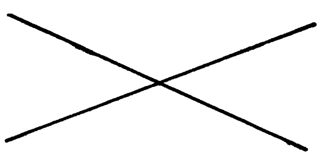
You will observe that this abridged form saves more than half the work and gives all of the former answer that was dependable.

By the abridged method, find the products to tenths:

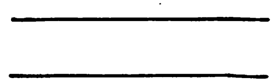
1. 34.6×38.42.
2. 9.362×84.52.
3. 62.96×3.1416.
4. 8.69×6.853.
5. 36.81×49.28.
6. 28.63×36.141.
7. 38.54×37.58.
8. 96.07×35.28.
9. 32.61×2.084.
10. 86.39×7.106.

8. PROPERTIES OF STRAIGHT LINES

Two straight lines that cross at a point are said to **intersect**. Two straight lines in the same plane that do not intersect are called **parallel**.



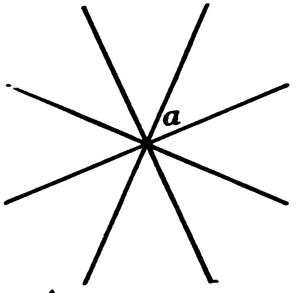
Intersecting Lines



Parallel Lines

1. Mark two points, A and B, on a sheet of paper. Using your ruler, draw a straight line that will pass through both points. Can you draw a second line through the same two points?

2. Mark a point on a sheet of paper. Draw two lines through this point. Can these two lines intersect at some other point?



3. Mark a point on a sheet of paper. Using your ruler, draw four straight lines through the point as in the figure. How many other straight lines can be drawn through this point?

4. Mark two points and measure the shortest distance between them.

From the above problems we see the following properties of straight lines:

1. *Any number of straight lines can be drawn through a point.*

2. *One and only one straight line can be drawn through two points; hence we say that two points determine a straight line.*

3. *Two straight lines can intersect in but one point; hence we say that two intersecting straight lines determine a point.*

4. *The length of the line segment connecting two points is the shortest distance between them.*

NOTE. — The segment connecting two points is taken to represent the distance between the points.

Exercises

1. Make a straightedge by folding and creasing a sheet of paper.

2. If the edge of your ruler is placed so as to touch two points of a straight line, will it touch the line all along the ruler? If it does not, what conclusion do we draw?

3. Draw a line along the edge of your ruler on a sheet of paper. Now reverse the ruler, and place it touching two points on the line you have drawn and draw a line. What will show whether the edge of the ruler is straight or not?

4. Explain why a gun or any sighting instrument has two "sights" on it.

5. Explain from the fourth principle above that a string stretched taut is straight.

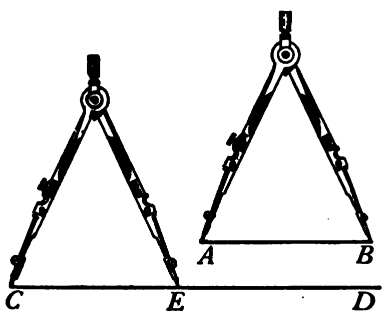
6. How could you locate a point known to be on each of two straight lines?

9. MEASURING AND COMPARING LINE SEGMENTS WITH COMPASSES

Two things are said to coincide when they are placed in identically the same position throughout.

Two line segments are equal if, and only if, they may be made to coincide.

Compasses may be used to mark off equal segments on a straight line, to construct a segment equal to a given segment, or to compare the length of two segments.

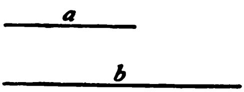


Thus, to mark off on CD a segment equal to AB, place one leg of the compasses at A. Then adjust the compasses so that the other point falls at B. Then turn the screw so as to hold the legs in this position. Then placing one leg at C, mark the point E on CD where the other point falls. Then $CE = AB$.

In the same way measure a segment by placing one point on a mark of your ruler and counting the distance to the mark where the other point falls.

1. Draw a line segment and measure it by use of compasses.

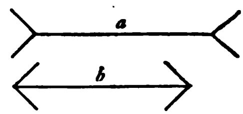
2. Draw two line segments of unequal length. Lay off on the longer a segment equal to the shorter.



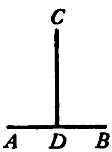
3. Draw a line segment twice as long as a given line segment.

4. Draw a line segment as long as the sum of segments a and b.

5. Lay off on b a segment equal to a, and thus show the difference in length between a and b.



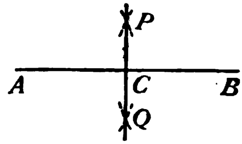
6. Estimate which of the two segments, a or b, shown in the margin, is longer, then check your estimate by use of the compasses.



7. Estimate which is longer, AB or CD, then with the compasses check your estimate.

10. HOW TO BISECT A LINE SEGMENT WITH RULER AND COMPASSES

Any line segment may be bisected as follows: Open the compasses so that the distance between the points is greater than half the length of the segment. Thus, to bisect AB open the compasses so that the distance between the points is more than half the distance from A to B. With the screw, clamp the arms so they will not change adjustment. Now with one point at A use the other to describe an arc. Then with one point at



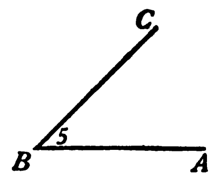
B describe an arc cutting the first arc in points *P* and *Q*. Connect *P* and *Q*, and the intersection *C* divides *AB* into two equal segments. Test the equality of *AC* and *CB* by use of compasses.

1. Draw three line segments and bisect each. Test the accuracy of your constructions by use of the compasses.

2. Divide a line segment into four equal segments by first bisecting it, and then bisecting each segment. By use of your compasses, see if the four segments are equal.

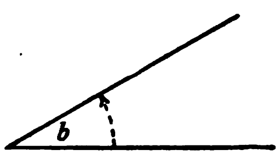
11. MEASUREMENT AND PROPERTIES OF ANGLES

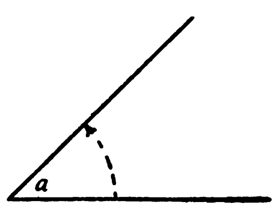
Two rays drawn from a point form an **angle**, thus, the rays *BA* and *BC* form an angle. The rays are called the **sides** of the angle, and the point *B* where they meet is called the **vertex** of the angle.



The symbol for angle is $\angle$. Thus, "angle *ABC*" is written $\angle ABC$. In reading or writing an angle expressed by three letters, the letter at the vertex is read or written in the middle.

Sometimes an angle is denoted by one letter written between the two sides as $\angle a$ and $\angle b$ shown below.





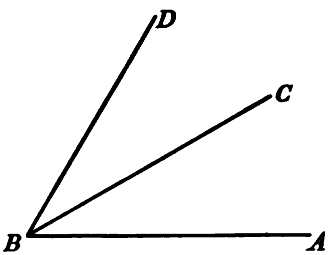
The size of an angle does not depend upon the length of the marks that represent the sides, but upon the relative direction of the sides.

The size, then, depends upon the amount of turning of one side about the vertex to make it fall upon the other. Thus, $\angle a$ is larger than $\angle b$, for one side would have to revolve farther to make it fall upon the other in $\angle a$ than in $\angle b$.

An angle, then, may be defined as the amount of turning made by a line rotating about a fixed point in a plane.

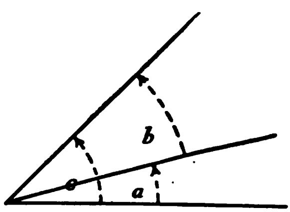
12. COMPARING TWO ANGLES

The sizes of two angles may be compared by placing one upon the other so that their vertices coincide, and so that a side of one falls upon a side of the other, and then observing the position of the other two sides. Thus, in the figure $\angle ABD$ is larger than $\angle ABC$.



In this figure we see that $\angle ABC + \angle CBD = \angle ABC$, and that $\angle ABD - \angle ABC = \angle CBD$.

Two angles are equal if, and only if, they may be made to coincide throughout. If an angle is placed upon an equal angle so that the vertices and a pair of sides coincide, the other sides must coincide.



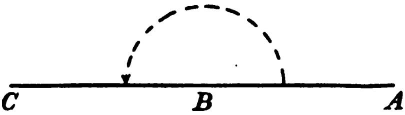
1. Draw two angles. Name one $\angle ABC$ and the other $\angle DEF$. Cut out or trace $\angle DEF$, and place upon $\angle ABC$ so that E coincides with B and so that ED falls upon BC. Which angle is larger? What angle shows the difference?

2. In the figure $\angle c = \angle a + \angle b$. If $\angle a = \angle b$, then $\angle c = 2 \angle a$ or $2 \angle b$.

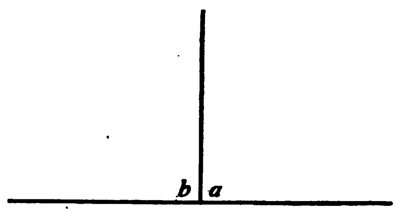
By tracing, make an angle equal to two given angles. In the same way make an angle equal to twice a given angle.

13. CLASSIFICATION OF ANGLES

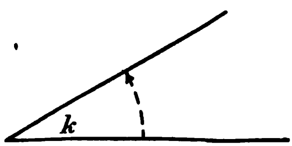
An angle whose sides extend in opposite directions, thus forming one straight line, is called a **straight angle.** Thus, $\angle ABC$ shown here is a straight angle.

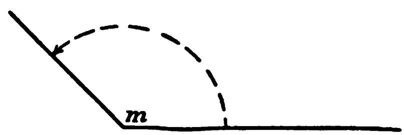


Half of a straight angle is a **right angle.** Thus, angles a and b shown here are *right angles.*



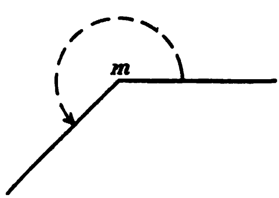
An angle less than a right angle, as angle k, is an **acute angle.** An angle

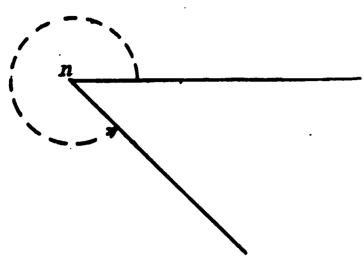




less than a straight angle, but greater than a right angle, as angle m, is an **obtuse angle.**

An angle greater than a straight angle, but less than two straight angles, as angles m and n shown below, are **reflex angles.**





14. THE MEASUREMENT OF ANGLES

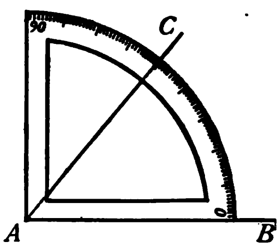
If a right angle is divided into 90 equal parts, each of these parts is called a **degree**, and is the principal *unit of measure* in measuring angles. Thus, a right angle is equal to 90 degrees, written 90°.

A degree is divided into 60 equal parts called minutes (written 60′), and a minute is divided into 60 equal parts called seconds (written 60″).

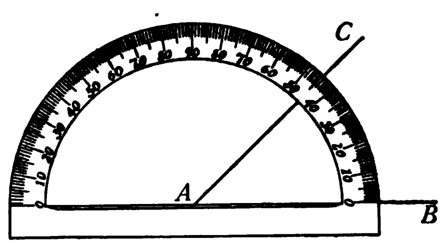
TABLE FOR ANGULAR MEASURE

180° = 1 straight angle
90° = 1 right angle
1° = 60′
1′ = 60″

The quadrant and the protractor are the instruments used for measuring the number of degrees in an angle. To use



A QUADRANT



A PROTRACTOR

either instrument for measuring angle BAC, the center is placed over the vertex A, as shown above. The instrument is then turned until the zero mark of the scale falls upon arm AB. Then the point where the arm AC crosses the scale shows the number of degrees in the angle.

1. Draw any angle and measure the number of degrees in it with a protractor.

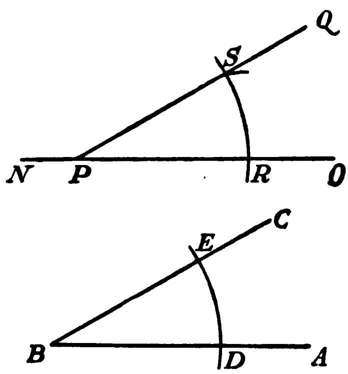
2. Make what seems to be a right angle, then measure it and see how nearly accurate you were.

3. Make what you think would be an angle of 60°, and then measure the angle you made to see how accurate you were.

4. Make what you think is an angle of 45°, and then measure it. How nearly accurate were you?

15. HOW TO CONSTRUCT AN ANGLE EQUAL TO A GIVEN ANGLE

An angle may be drawn equal to a given angle as follows: To draw an angle equal to $\angle ABC$, first draw a straight line as NO. Now with one leg of the compasses at B, the vertex of the given angle, describe an arc cutting the side BA and BC in D and E, respectively. With the same opening, and one leg at some point P on line NO, describe an arc cutting NO at R. Next, place the point of one leg of the compasses at D and adjust the compasses so the other leg will fall at E. Now with the same opening, place one point at R and describe an arc cutting the arc already described at S. Draw a straight line from P through S and $\angle OPQ = \angle ABC$. Test the equality of $\measuredangle$ OPQ and ABC by use of a protractor.



1. Draw any angle and construct a second angle equal to it. By use of the protractor test the accuracy of your construction.

2. Draw any two angles and then construct an angle equal to their sum. Test your construction by the use of a protractor.

3. Draw any two angles and then construct one twice as large as their sum. Test the accuracy of your construction by the use of a protractor.

4. Draw two angles and then construct an angle equal to their difference. Test the accuracy of your construction as in problem 3.

5. Make any angle. Estimate its size and then measure it to see how accurately you estimated.

6. Draw any angle. Then draw an angle three times as large. Test your accuracy by the use of a protractor.

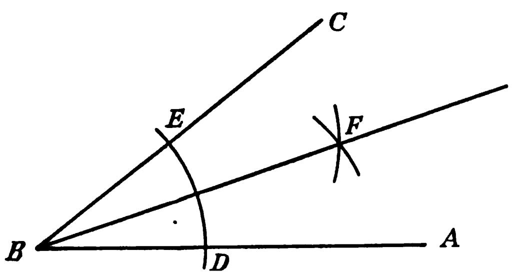
7. Use your protractor to draw angles of 15°, 30°, 45°, 60°, 75°, and 120°.

8. Make an angle that you think is 40°, and then measure it to see how nearly correct you were.

9. Draw angles of any size as your teacher may direct and measure them to see how nearly correct you were.

16. HOW TO BISECT AN ANGLE

Any angle may be **bisected** (divided into two equal parts) by the use of compasses and a straightedge. The method is here shown. With one arm of the compasses at B, describe an arc cutting arms BA and BC in D and E. Now with D and E as centers, describe arcs intersecting in F. Draw BF and $\angle ABF = \angle FBC$.



1. Draw any angle and bisect it. Test the accuracy of your construction by the use of a protractor.

2. Draw an angle nearly 90° and divide it into four equal parts. Test the accuracy of your construction by the use of a protractor.

3. Draw two intersecting lines and bisect each of the four angles.

4. Draw other intersecting lines and bisect the four angles thus formed and see if the same thing seems true of the bisectors. That is, see if the bisectors are perpendicular to each other.

17. COMPLEMENTARY, SUPPLEMENTARY, AND VERTICAL ANGLES

It is often necessary to refer to two angles whose sum is equal to 90°, or a right angle, as angles *a* and *b*. Such angles are called **complementary**, or one is said to be the **complement** of the other.

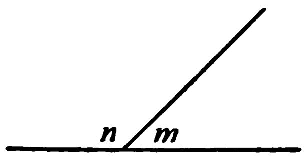
1. What is the complement of an angle 60°? Of 45°? Of 72°? Of 80°? Of 30°?

2. By the use of compasses and a straightedge, construct the complement of a given angle. That is, make any acute angle and then construct an angle which with it makes 90°.

3. By the use of a protractor and a ruler construct the complement of an angle of 35°.

When the sum of two angles equals 180° or a straight angle, one is called the **supplement** of the other, or they are

called **supplementary**. Thus, in the figure, angle m is the supplement of angle n.



4. What is the supplement of an angle of 80°? Of 90°? Of 120°? Of 115°?

5. Show how with a ruler alone you can construct the supplement of a given angle.

6. What angle is 5 times its supplement?

SUGGESTION. — Form an equation. Let x equal the smaller of the two angles.

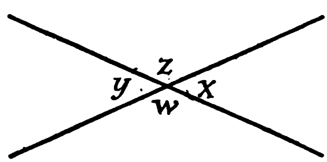
7. Two angles are complementary. One is 2 times the other. Find each.

8. One of two complementary angles is 10° larger than the other. Find each.

9. When one of two supplementary angles is 40° larger than the other, what is the size of each?

10. Two times one angle is 10° more than its complement. Find the angle and its complement.

When two straight lines intersect, they form four angles as shown in the margin. Of these, any two not adjacent are called **vertical angles**. Thus, x and y, and z and w are two pairs of vertical angles.



11. What should you judge to be true of the sizes, compared with each other, of any two angles that are vertical angles?

12. With your compasses, measure each pair of vertical angles and compare them. That is, how does the size of x compare with y?

13. In the figure shown how large is $x+w$? How large is $y+w$?

Since $\qquad x+w=180°$ and $y+w=180°$,

it follows that $\qquad x+w=y+w$.

Subtracting w from each member,

$$x=y.$$

Thus we have proved through a course of reasoning that vertical angles are equal to each other.

18. THE EQUALITY OF SPECIAL ANGLES

In much of the work of mathematics there are certain facts that we should know just as we know the multiplication tables in arithmetic, for we use the facts so often in certain matters that arise.

Thus, since all straight angles are equal to 180°,

1. *Any two straight angles are equal.*

Since all right angles are equal to 90°,

2. *Any two right angles are equal.*

If $\angle a$ is the complement of $\angle b$, and $\angle c$ is the complement of $\angle b$, then

$$\angle a+\angle b=\angle c+\angle b,$$

for each pair equals a right angle.

Subtracting $\angle b$ from each member,

$$\angle a=\angle c.$$

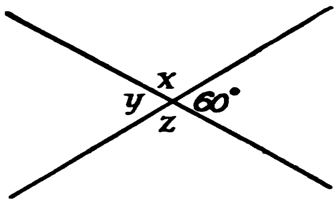
That is,

3. *The complements of equal angles are equal.*

In the same way show that

4. ***The supplements of equal angles are equal.***

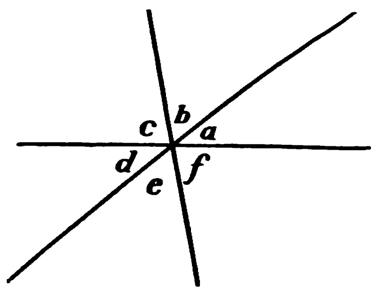
Read problem 13, page 51, and show that



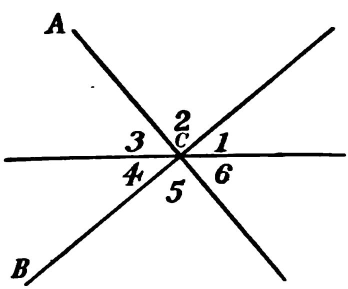
5. ***Vertical angles are equal.***

1. In the figure given in the margin tell at sight how many degrees in each angle and tell how you know.

2. If one of the four angles made by two intersecting lines is 90°, what do you know about the others? Why?

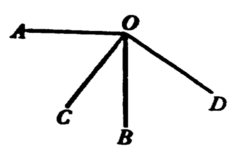


3. If three straight lines intersect as in the margin, how many of the six angles should you have to measure in order to know the size of all? If $\angle a = 40°$ and $\angle b = 60°$, tell the size of each of the others.



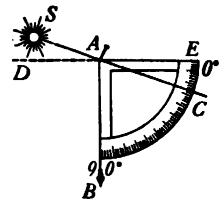
4. In the figure in the margin $\angle 1 = 40°$ and $\angle 3 = 50°$. How large is angle ACB? Explain how you know.

5. In the adjoining figure, $\angle AOB$ and $\angle COD$ are right angles. Explain why $\angle AOC$ must equal $\angle BOD$.

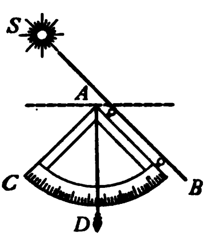


6. The angle of elevation of the sun above the horizon, as $\angle SAD$, may be measured as follows:

A quadrant is held in a vertical position, so that a plumb line AB, which is suspended from a pin at A, falls upon the 90° mark. The shadow of the pin falls upon the scale at C, which shows the angle of elevation of the sun. Explain why.

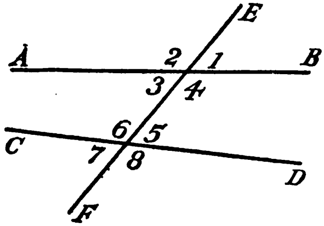


7. Tycho Brahe (1546–1601), a Danish noble who built and operated the first real astronomical observatory, contructed and used a big quadrant for measuring the altitudes of the stars, or their angles of elevation above the horizon. When the instrument was held in a vertical position and the sights A and B aligned with the star S, the angle of elevation of the star was determined by reading $\angle\ CAD$. Explain why.



19. PARALLELS AND TRANSVERSALS

When a straight line intersects two or more straight lines it is called a **transversal** of these lines. A transversal of two lines makes four angles at each intersection. These eight angles are so important when the two intersected lines are parallel that they are named as follows:



Angles 3, 4, 5, and 6 are **interior angles.**

Angles 1, 2, 7, and 8 are **exterior angles.**

Such angles as 1 and 5 are called **corresponding angles.**

The pairs 3 and 5, and 4 and 6 are **alternate interior angles.**

The pairs 2 and 8, and 1 and 7 are **alternate exterior angles.**

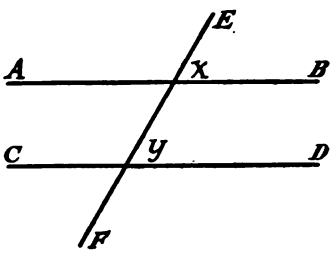
The pairs 4 and 5, and 3 and 6 are **consecutive interior angles.**

The pairs 1 and 8, and 2 and 7 are **consecutive exterior angles.**

When the intersected lines are parallel, there are certain pairs whose angles are equal. These will be shown in what follows.

20. COMPARING CORRESPONDING ANGLES

Carefully draw two parallel lines by use of your ruler or trace two parallel lines on ruled or squared paper and draw a transversal.



Compare any two of the corresponding angles by use of a protractor. Draw a new figure and make similar measurements. What is your conclusion?

If carefully drawn and carefully measured, you found that

If two parallel lines are cut by a transversal, the corresponding angles are equal.

Establishing Truth by Reasoning

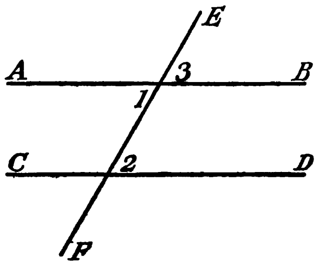
The geometric truth discovered above was done by actual measurement. This method of discovering truths is a slow, and often unsafe, method of establishing general truths. It is better when possible to arrive at general truths through a process of reasoning based upon known facts.

From the facts already known from measurement many

new facts can be established through a course of reasoning, as will be shown in the proofs that follow.

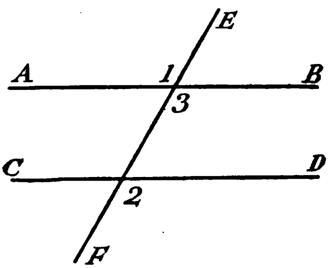
NOTE. — The first four facts proved are important and should be remembered.

1. *Prove that if two parallel lines are cut by a transversal, the alternate interior angles are equal.*



PROOF: $\angle 1 = \angle 3$, for they are vertical angles. And $\angle 2 = \angle 3$, for they are corresponding angles. Then $\angle 1 = \angle 2$, for they both equal $\angle 3$.

The last statement is based upon the evident fact, called an **axiom**, that

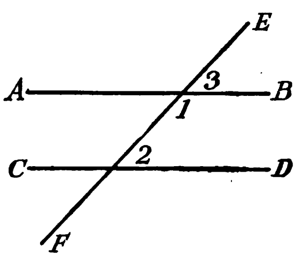


Things which are equal to the same thing, or to equal things, are equal to each other.

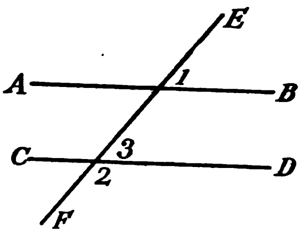
2. *Prove that if two parallel lines are cut by a transversal, the alternate exterior angles are equal.*

SUGGESTION. — Find angles that you know to be equal and follow the line of proof given above. That is, prove that $\angle 1$ and $\angle 2$ are each equal to $\angle 3$.

3. *Prove that if two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.*



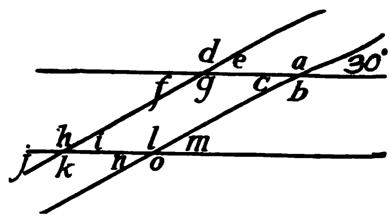
SUGGESTION. — You are to prove that $\angle 1 + \angle 2 = 180°$. What do you know about $\angle 1 + \angle 3$? What do you know about $\angle$s 2 and 3?



4. *Prove that if two parallel lines are cut by a transversal, the consecutive exterior angles are supplementary.*

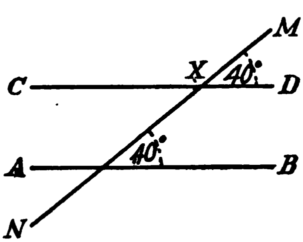
SUGGESTION. — Prove $\angle 1 + \angle 2 = 180°$. What do you know about $\angle 2 + \angle 3$? About $\angle$s 3 and 1? What is your conclusion?

5. In the figure in the margin, two parallel lines are cut by two parallel transversals making one angle 30°, as shown. Tell, without measuring, the size of all the other angles formed.



21. HOW TO DRAW A LINE THROUGH A GIVEN POINT PARALLEL TO A GIVEN LINE

You found by measurement that if two parallel lines are cut by a transversal, the corresponding angles are equal.

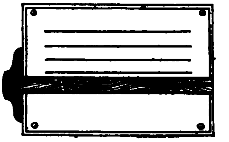


Now suppose that line MN cuts line AB, making an angle of 40°. Select some point on MN as point X and with XM as one side, make an angle of 40°. Produce AB and CD as far as your paper will allow. Make similar constructions for other angles. What is your conclusion?

What *seems* to be true from measurement and construction is proved in geometry by a course of reasoning to be true. That is,

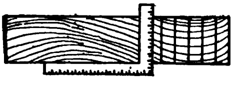
If two straight lines are cut by a transversal making the corresponding angles equal, the lines are parallel.

Since two right angles are equal, it follows that



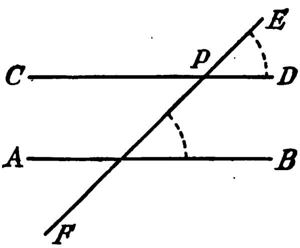
If two straight lines are perpendicular to the same straight line, they are parallel.

1. In using the T-square to rule parallel lines, what principle is used?

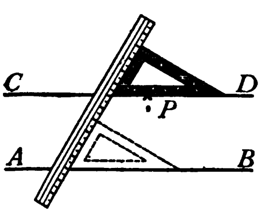


2. When a carpenter marks off parallel lines with a carpenter's square as shown in the figure, what principle is he using?

3. Using either of the two principles given, by use of straightedge and compasses, draw through a given point a line parallel to a given line.



SUGGESTION. — Through P draw EF cutting AB. With P as a vertex and PE a side, construct an angle equal to the angle made by EF and AB.



4. Study the figure in the margin and show how by use of a *draftsman's triangle* and a ruler, a line may be drawn through P and parallel to AB.

5. Draw through a point a line parallel to a given line, by use of straightedge and compasses, as in problem 3, until you can do it accurately and with ease.

6. By use of a ruler and a triangle, as in problem 4, draw through a given point a line parallel to a given line. Repeat the construction until proficient.

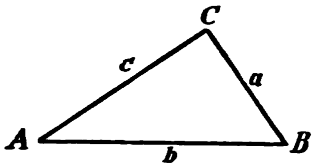
7. Make a large triangle of wood and use it in making parallel lines on the blackboard.

CHAPTER IV

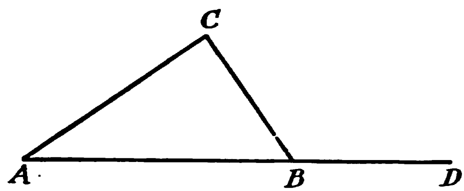
PROPERTIES AND USE OF THE TRIANGLE

1. THE NOTATION OF TRIANGLES

When three points not in a straight line are connected by three line segments, the figure formed is called a **triangle**.



The three points are called the **vertices** of the triangle, and the line segments are called the **sides**. Thus, the figure shown here is called a triangle ABC, or $\triangle\ ABC$. A, B, and C are the *vertices*. The sides opposite the vertices are called a, b, and c, respectively. The sides are also read AB, BC, and CA. The three angles formed by the sides, as angle ABC, are called the **interior** angles of the triangle. They are usually denoted by the letter at the vertex only, as $\angle A$, $\angle B$, $\angle C$.



When a side is extended through a vertex, as AB extended to D, the angle DBC is called the **exterior** angle at B.

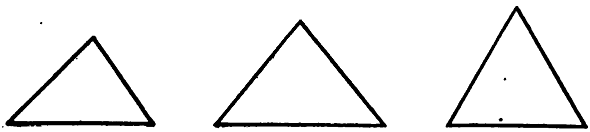
2. TRIANGLES CLASSIFIED AS TO SIDES

In referring to triangles it is sometimes convenient to classify them as to sides.

A **scalene** triangle is one that has *no two* sides equal.

An **isosceles** triangle is one that has *two* sides equal.

An **equilateral** triangle is one that has *all three* sides equal.



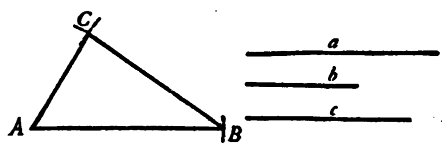
SCALENE ISOSCELES EQUILATERAL

In an isosceles triangle the vertex formed by the two equal sides is called the **vertex** of the triangle and the other side is its **base.**

In an isosceles triangle the base angles are equal.

1. By use of compasses and a straightedge construct a triangle whose sides are three given segments.

SUGGESTION. — Draw a straight line and lay off $AB = c$. Then from A describe an arc with radius b, and from B describe an arc with radius a and let them intersect at C.



2. Construct an isosceles triangle with a given base and given sides.

3. Construct an isosceles triangle each of whose equal sides is twice the base.

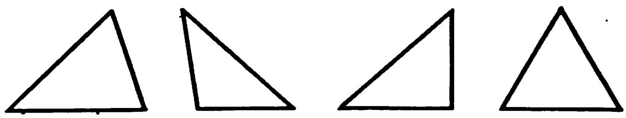
4. Construct an equilateral triangle each of whose sides is a given length.

3. TRIANGLES CLASSIFIED AS TO ANGLES

Triangles are often classified as to angles as follows:

An **acute** triangle is a triangle having all its angles acute.
An **obtuse** triangle is a triangle having one obtuse angle.
A **right** triangle is a triangle having one right angle.
An **equiangular** triangle has all of its angles equal.

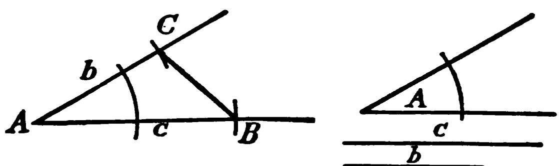
An equiangular triangle is also equilateral.



ACUTE OBTUSE RIGHT EQUIANGULAR

In a right triangle the side opposite the right angle is called the **hypotenuse** and the other two sides are called the **legs**, of the triangle.

1. Construct a triangle having two given sides and an angle equal to a given angle.



SUGGESTION. — Lay off a line equal to c. Now with c as one of its sides, construct an angle equal to A. Lay off on AC a segment equal to b.

2. Construct a right triangle with its legs equal to two given segments, say 2″ and 3″.

3. Construct an isosceles right triangle with each of its legs equal to a given segment, say 2″.

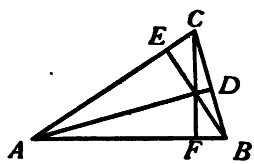
4. Construct a right triangle with one leg and the hypotenuse a given length. (The hypotenuse must be longer than the leg.)

5. Construct a right triangle with the hypotenuse twice as long as one of its legs.

4. THE ALTITUDES AND MEDIANS OF A TRIANGLE

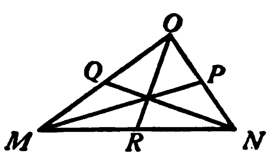
The perpendicular from any vertex of a triangle to the opposite side is called an **altitude** of the triangle, and the side to which it is drawn is called the corresponding **base.**

A line segment joining any vertex of a triangle with the middle point of the opposite side is called a **median** of the triangle.



1. How many altitudes has a triangle?

2. Draw any acute triangle and carefully draw its three altitudes.



3. Draw other acute triangles and construct their altitudes. Do the altitudes seem to meet in the same point? Do they seem to meet within or without the triangle?

4. Draw an obtuse triangle. Draw the altitude from one of the acute angles.

SUGGESTION. — The side opposite the acute angle must be produced. The altitude is said to meet the opposite side *on the side produced.*

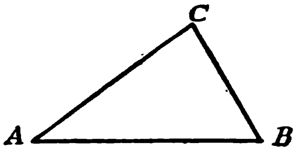
5. Draw a right triangle. From either acute angle, draw an altitude.

6. When all the altitudes fall within the triangle, what kind of triangle is it?

7. When an altitude coincides with one of its sides, what kind of triangle is it?

8. When an altitude meets the opposite side produced through a vertex, what kind of triangle is it?

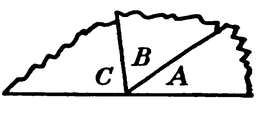
5. MEASURING THE INTERIOR ANGLES OF A TRIANGLE



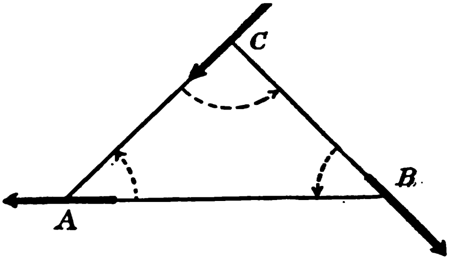
1. Draw a triangle, ABC, and carefully measure each of its angles with a protractor. Add the three angles and find the sum.

2. Draw at least three other triangles and find the sum of the three angles.

3. Draw a triangle on paper or cardboard. Now tear off the corners and place the three angles adjacent as in the figure. What seems to be the size of the angle formed by all three angles? Test your answer by use of a straight-edge.



4. Draw a large triangle whose sides are from 6 to 10 inches and measure as follows: Lay a pencil along AB as in the figure. Then rotate the pencil through angle A as indicated by the arrow. Then move along AC to the position at C and again rotate through angle C as shown by the arrow. Finally move along CB



to the position shown and rotate through angle B. Compare the directions of the pencil with its original position and see what part of a complete turn it has made. Then, again, the sum of the three angles seem to be equal to how many degrees?

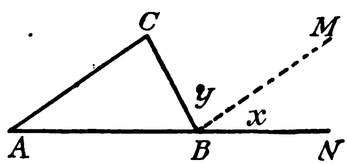
From the three methods used you have perhaps inferred the following important truth, that

The sum of the three angles of any triangle is equal to 180°.

This is one of the most important truths of geometry. Thus, if one knows two angles of any triangle, the third angle is known. Or, if one acute angle of a right triangle is known, all the angles are known.

5. From facts that you know from the last chapter, you could have proved the fact given above, through a course of reasoning.

Suppose BM is drawn parallel to AC, what do you know about angles A and x? What do you know about angles C and y? Since ABN is a straight line, what do you know about the sum of angles B, y, and x? Since $\angle x + \angle B + \angle y = 180°$, what do you know of $\angle A + \angle B + \angle C$? You have thus established through a course of reasoning the fact inferred from measurement.



6. If the three angles of a triangle are equal, what must be the size of each?

7. If one angle of a triangle is 50°, and another 60°, what is the third?

8. If one angle of a right triangle is 30°, what is the size of the other acute angle?

9. If one angle of a triangle is a right angle, why must the other two angles be complementary?

10. If the three angles of a triangle are x, $2x$, and $3x$, how many degrees in each? (Form an equation.)

11. In a certain right triangle, one acute angle is four times the other. What is the size of each?

12. If two angles of a triangle are each equal to 50°, how large is the third?

13. One acute angle of a right triangle is 20° larger than the other. What is the size of each?

14. In triangle ABC, $\angle A$ is 15° larger than $\angle B$ and 20° smaller than $\angle C$. Find the size of each.

15. In a certain triangle one angle is twice the smaller and the other six times the smaller. What is the size of each?

16. One acute angle of a right triangle is four times the other. What is the size of each?

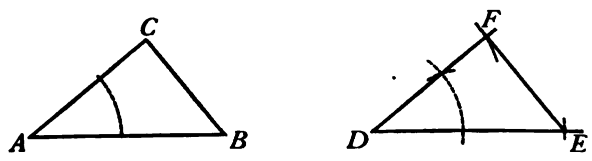
17. If the vertical angle of an isosceles triangle is 80°, what is the size of each of the equal angles (called *base angles*)?

6. THE CONGRUENCE OF TRIANGLES

If one of two figures may be placed upon the other so that they coincide throughout, the figures are called **congruent**.

1. Draw upon cardboard or paper a large triangle ABC. Construct a second triangle DEF so that $DE = AB$, $DF = AC$, and $\angle D = \angle A$.

SUGGESTION. — Study the figure and see how $\triangle DEF$ was constructed.



2. With shears, cut out $\triangle DEF$, which you constructed in problem 1, and place it upon $\triangle ABC$ and observe that they may be made to coincide.

The general truth observed in problem 2 is,

If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the two triangles are congruent.

3. Draw a triangle ABC. Draw on cardboard or paper a second triangle DEF with $DE = AB$, $\angle D = \angle A$, and $\angle E = \angle B$. Cut out $\triangle DEF$ and apply it to $\triangle ABC$. What seems to be true?

The general truth of what seems to be true in problem 3 is,

If two angles and the included side of one triangle are equal respectively to two angles and the included side of another, the two triangles are congruent.

4. Draw any triangle ABC. On cardboard or paper construct a triangle DEF with $DE = AB$, $EF = BC$, and $DF = AC$. Cut out $\triangle DEF$ and apply it to $\triangle ABC$. What seems to be true?

The general truth of what seems to be true in problem 4 is,

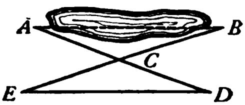
If three sides of one triangle are equal respectively to the three sides of another, the two triangles are congruent.

The three important truths that you have just learned about congruent triangles are proved in geometry by a course of reasoning. They are among the most important truths of geometry.

7. DISTANCES AND ANGLES COMPARED BY CONGRUENCE OF TRIANGLES

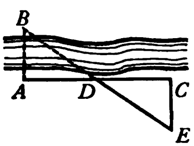
Since the corresponding parts of congruent triangles are equal, angles and distances can be measured by means of the truths just learned.

1. In order to find the distance from A to B across a lake, surveyors measured off a straight line AC and extended it to D so that $CD = AC$. Then they measured the distance BC and extended BC to E so that $CE = BC$. Then they measured the distance from E to D. Prove that AB equals the measured distance DE.

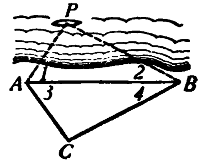


SUGGESTION. — Prove the triangles congruent.

2. The distance AB across a stream may be found as follows: Measure AC at right angles to AB. Locate a point D halfway between A and C. Measure CE at right angles to AC, to a point E in line with B and D. Prove that AB equals CE.

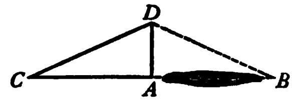


3. The Greeks as early as the time of Thales (640–546 B.C.) determined the distance AP of a ship from shore by means of the congruence of triangles having two angles and the included side of one equal respectively to two angles and the included side of the other. They measured $\angle 1$ and $\angle 2$. Explain how they were then able to mark off a distance on the shore equal to AP and thus find the distance AP.

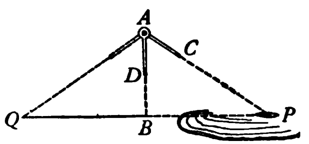


4. The distance from A to an inaccessible point B may be found approximately without instruments as follows: Stand

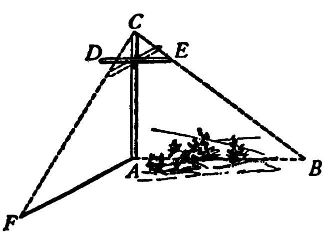
at A and look at B, raising or lowering the head until your hatbrim falls in the line of sight. Then, without raising or lowering the head, turn about, and observe the point C at which the line of sight strikes the ground. Pace the distance from A to C. In the diagram, D represents the position of the eyes. Prove that $AB = AC$.



5. Thales (problem 3) is said to have made an instrument for determining the distance BP of a ship from shore. It consisted of two rods AC and AD, hinged together at A. Rod AD was held vertically over point B, while rod AC was pointed toward P. Then, without changing $\angle DAC$, the instrument was revolved about AD, and point Q noted on the ground at which the arm AC was directed. BQ was then measured. Prove that $BP = BQ$.

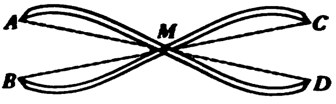


6. An instrument used as late as the sixteenth century for finding the distance from A to an inaccessible point B was called a *cross staff*. It consisted of a vertical staff AC to which was attached a horizontal crossbar DE that could be moved up or down on the staff. Sighting from C to B, DE was lowered or raised until C, E, and B were in a straight line. Then the instrument was revolved about CA and the point F at which the line of sight CE met the ground marked. The distance AF was then measured. Prove that $AB = AF$.

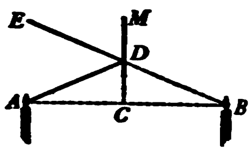


NOTE. — Students may easily make and use this instrument.

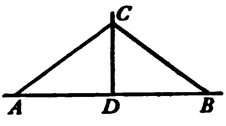
7. The calipers shown in the drawing may be used for measuring the thickness of objects, such as the diameter of pipes, the diameter of trees in forestry, etc. AD and BC are pivoted at M. Points A, M, and D are in a straight line. M is the middle point of both AD and BC. Prove that the distance between A and B equals the distance between C and D. Hence explain how the instrument may be used.



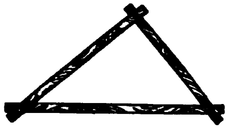
8. Every one is familiar with the fact that if an object is placed before a plane mirror, its image appears to be as far behind the mirror as the object is in front of it. M is an edge view of a mirror. Light from an object at A strikes the mirror at D and is reflected to the eye at E. The mind projects the line ED through the mirror to B, forming the image at B. It is known from science that $\angle MDE = \angle ADC$ and that $CD \perp AB$ (*i.e.* CD is perpendicular to AB). Prove that $CB = AC$.

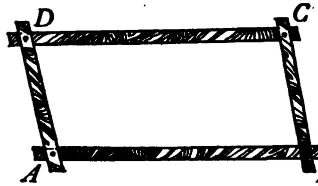


9. If two oblique line segments drawn from a point on a perpendicular to a line cut off equal distances on the line from the foot of the perpendicular, prove that the oblique segments are equal.



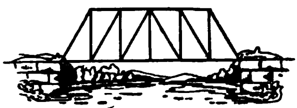
10. Nail three strips of wood together so as to form a triangle, using only one nail at each joint. Is this frame rigid, or can it be changed into different shapes by exerting pressure upon it? Through what principle of the congruence of triangles would you have inferred the result?



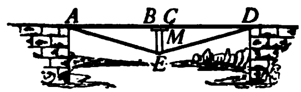


11. In the same way nail four strips of wood together as shown in the drawing. Can this frame be changed into different shapes by exerting pressure upon it?

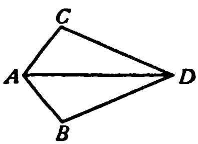
12. Why is a roof sufficiently braced when a board is nailed across each pair of rafters?



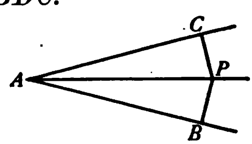
13. Why is a long span of a bridge in which the truss is made with queen posts and diagonal rods, as shown in the drawing, sufficiently supported?



14. The drawing shows a piece of a bridge called an inverted king-post truss. AB and CD are steel bars resting on a plate M. The plate M is supported by wire ropes EA and ED. Show why this structure will support a heavy weight placed upon it.

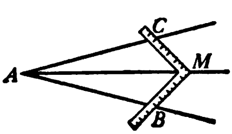


15. In the kite shown in the figure, $AB = AC$ and $BD = CD$. Prove that AD bisects $\angle BAC$ and $\angle BDC$.



16. In the angle BAC, $PC \perp AC$, $PB \perp AB$, and $PC = PB$. Prove that AP bisects $\angle BAC$.

17. A carpenter bisects an angle BAC with his steel square as follows: He marks off AC and AB of equal length on sides of the angle. Then he places his square so that $MC = MB$, as shown in the drawing. Then he marks the point M at the heel of the square, and draws AM. Prove that AM bisects $\angle BAC$.



CHAPTER V

POSITIVE AND NEGATIVE NUMBERS: ADDITION AND SUBTRACTION

IN mathematics we need to use numbers *opposite* in nature. That is, numbers such that if one means a gain, the other means a loss; or if one number means a temperature above zero, the other means a temperature below zero; or if one means a distance east of a given point, the other means a distance west of it; etc.

1. OPPOSITE NUMBERS IN SCORING GAMES

You have no doubt played games in which you scored a certain amount if you won and "went in the hole" if you lost. Thus, beginning with nothing or a "zero score" you may have missed and recorded "2 in the hole," thus 2. The next trial you may have scored 3 and then you counted off 2 and had a score of 1. Or you may have "gone in the hole 5." Then scoring 2, you were still "3 in the hole." Thus you see that you made *two kinds of scores opposite* in nature.

One kind added to your final score or increased it; the other was taken from your final score or lessened it. Any score increasing your final score might have been called a *positive* score and the other a *negative* score.

Thus a score adding 5 to the total could have been written +5, while "5 in the hole" could have been written −5. These are called signed numbers.

WHO WON?

	FRANK	JOHN
1st score	3	2
2nd score	2	2
3rd score	4	3
4th score	2	2
5th score	1	1
	4	2

WHO WON?

FRANK	JOHN
+3	+2
−2	+2
+4	−3
−2	+2
+1	−1
+4	2

WHO WON?

HELEN	MARY
5	2
2	4
4	3
3	2
1	1
1	0

WHO WON?

HELEN	MARY
−5	2
2	−4
4	3
−3	−2
1	1
−1	0

NOTE.—When no sign is written before a number it is considered positive.

If these represent scores made in five trials find the final score:

1.	2.	3.	4.	5.	6.	7.	8.
3	−2	8	9	−5	6	−1	6
−4	3	−5	−6	−3	2	3	−3
−2	2	−3	5	8	−5	−5	4
5	−4	4	−4	2	−3	4	2
1	2	2	−3	−4	4	2	−1

9. Tell how you found the final scores in the exercises given above.

Find the final score:

10.	**11.**	**12.**	**13.**	**14.**	**15.**	**16.**	**17.**
− 3	4	− 5	− 3	6	− 8	7	− 8
− 2	2	3	7	3	− 2	− 3	4

18.	**19.**	**20.**	**21.**	**22.**	**23.**	**24.**	**25.**
− 3	4	− 5	− 2	4	− 4	5	− 5
− 1	2	3	− 4	3	− 6	− 2	− 6
− 5	6	7	6	− 5	7	4	7

The number regardless of the sign is called the **absolute value** *or* **numerical value.**

26. In exercise 10, explain how you got − 5 for the final score.

27. In exercise 11, explain how you got 6 for the final score.

28. In exercise 12, explain how you got − 2 for the final score.

29. In exercise 13, explain how you got 4 for the final score.

Observe, then, that you found the scores as follows:

1. *When the signs were alike, you added the absolute values and prefixed the sign of the addends to the sum.*

2. *When the signs of the two addends were unlike, you found the difference between the two addends and prefixed the sign of the larger.*

This is called **algebraic addition.**

Find the final scores:

30.	31.	32.	33.	34.	35.	36.	37.
4	−8	9	−6	3	−4	9	−3
−5	6	−3	−2	8	8	3	−8
6	−2	−2	5	−5	−6	−7	6
−3	3	4	4	−2	−7	6	−2

38. Tell how you found each final score.

Observe, then, that in adding signed numbers you took the following steps:

1. *Add the positive and negative numbers separately.*
2. *Find the difference between the absolute values of the sums.*
3. *Prefix the sign of the greater of the two sums.*

Add:

39.	40.	41.	42.	43.	44.	45.
28	−42	48	53	−72	−84	34
−36	96	56	−18	−36	28	−92
−42	78	−32	−47	−28	−36	−75
34	−64	−45	93	42	53	−19
96	−87	96	85	75	−86	46

2. A FURTHER MEANING OF NEGATIVE NUMBERS

In the scoring games that you have considered, you saw that a negative score was to be subtracted. Thus, a score of 5 and −3 gave a final score of 2; and a score of 3 and −5 gave a final score of −2. You saw, too, that if you have a score of −2 and a positive score is made, 2 must be taken from it. Thus you see two new meanings of a negative number. That is,

1. *A negative number is in nature a subtrahend.*

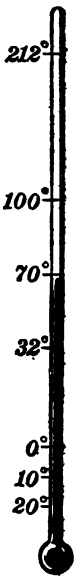
2. *A negative number indicates a reserved subtraction.*

A negative number, then, makes subtraction always possible. Without it, the minuend must always be larger than the subtrahend.

Thus, in the formula $A = b - c$, when $b = 6$ and $c = 2$, $A = 4$. But when $b = 2$ and $c = 6$, $A = -4$. That is, to take 6 from 2 is only possible by taking 2 from 2 and having -4 left as a *reserved subtraction.*

Thus, if you have \$6 and owe \$2 you are yet worth \$4, for \$6 − \$2 = \$4; if you have \$6 and owe \$6, you are worth \$0 or nothing, for \$6 − \$6 = \$0; but if you have \$6 and owe \$10, you could pay but \$6 of it and still owe \$4. That is, you are worth −\$4. This is represented by

\$6 − \$10 = −\$4.



3. OPPOSITE NUMBERS USED TO REPRESENT TEMPERATURE

Temperature is often represented by opposite numbers, "above zero" being represented by a positive number, and "below zero" by a negative number.

1. At 8 A.M. the temperature in a certain town was $+3°$. At 1 P.M. it was $+8°$. How many degrees did the temperature rise?

2. If it is $+4°$ at 7 A.M. and rises $5°$ during the day, what will represent the temperature at the end of the day?

3. If the temperature is $+3°$ at 6 P.M. and falls $12°$ during the night, what will then represent the temperature?

4. If the temperature is $-3°$, what will it be when there is a rise of $10°$? When there is a fall of $10°$?

Find the difference in temperature:

5. $-3°$ and $+4°$.
6. $-3°$ and $-8°$.
7. $+2°$ and $-10°$.
8. $-5°$ and $+14°$.
9. $+8°$ and $-3°$.
10. $-10°$ and $-2°$.
11. $+4°$ and $-4°$.
12. $-3°$ and $-8°$.

Give the rise in temperature:

13. From $+3°$ to $+8°$.
14. From $-2°$ to $+5°$.
15. From $-8°$ to $-2°$.
16. From $-20°$ to $-5°$.
17. From $-5°$ to $+15°$.
18. From $+2°$ to $+12°$.

Give the fall in temperature:

19. From $+2°$ to $-5°$.
20. From $-3°$ to $-10°$.
21. From $+3°$ to $-1°$.
22. From $+18°$ to $+12°$.
23. From $+4°$ to $-4°$.
24. From $-3°$ to $-18°$.

Letting + denote a rise in temperature, exercises 13–18 may be written:

$$\begin{array}{rrrrrr}
+8° & +5° & -2° & -5° & +15° & +12° \\
+3° & -2° & -8° & -20° & -5° & +2° \\
\hline
+5° & +7° & +6° & +15° & +20° & +10°
\end{array}$$

Letting − denote a fall in temperature, exercises 19–24 may be written:

$$\begin{array}{rrrrrr}
-5° & -10° & -1° & +12° & -4° & -18° \\
+2° & -3° & +3° & +18° & +4° & -3° \\
\hline
-7° & -7° & -4° & -6° & -8° & -15°
\end{array}$$

If we use the same terms used in arithmetic, calling finding a difference **subtraction**, the number from which we begin to count the **subtrahend** and the one to which we count the **minuend**, the problems in temperature illustrate **algebraic subtraction.**

NOTE. — The lower number or subtrahend is the one from which we count. Use + to denote a rise and − to denote a fall.

Subtract as in the preceding exercises:

25.	26.	27.	28.	29.	30.	31.
$+10°$	$+8°$	$-12°$	$-15°$	$+12°$	$-5°$	$+2°$
$+5°$	$-3°$	$-4°$	$+3°$	$-5°$	$-17°$	$-10°$

Observe exercises 25 to 31 and notice an illustration of the law that

Algebraic subtraction is changed into algebraic addition by changing the sign of the subtrahend.

Subtract:

32.	33.	34.	35.	36.	37.	38.
$+54$	-38	$+21$	-35	-12	$+17$	-12
$+19$	-16	-38	$+16$	$+42$	-16	-54

4. SIGNED NUMBERS USED IN STOCK QUOTATIONS

If you will refer to the stock quotations of a daily paper you will notice that the "net change" in the price of stock is indicated by plus and minus signs. A glance at the right-hand column shows the extent of the rise or fall of a certain stock from the preceding day.

NEW YORK STOCK DEALINGS

Fluctuations on July 10, 1920.

Shares	High	Low	Close	Net Chge.	Shares	High	Low	Close	Net Chge.
200 Ajax Rubber	65¼	65¼	65¼	−2⅜	2600 Baltimore & O.	33⅛	32⅜	33⅛	+ ¼
100 Allis-Chalmers	88½	88½	88½	− ⅜	6100 Canadian Pacific	122¾	118⅛	121¾	+2⅞
800 Am. Beet Sugar	94	94	94	− ¼	200 Chi. Pneu. Tool	98	97	97	−3½
1600 Am. Locomotive	102½	102	102	− ⅛	1200 Chi., M. & St. P.	35	34¾	35	+ ¾
500 Am. Ship & C.	24¾	25⅝	24¾	− ¼	1700 Columbia Graph.	31⅝	31	31	− ¼
400 Am. Sm. & R.	62	61¾	62	− ⅛	800 Gen. Electric	144	144	144	+ ¼
13100 Am. Woolen	95½	90¼	91½	−4⅛	12000 Gen. Motors	27⅛	26⅜	27	0
8000 Baldwin Loco.	124⅝	123¾	123⅞	− ⅝	17800 Invisible Oil	44⅞	42⅛	44⅞	+3

1. Which stocks rose, and how much, over the price of the preceding day?

2. Which stocks fell, and how much?

5. THE USE OF SIGNED NUMBERS TO DISTINGUISH BETWEEN OPPOSITE DIRECTIONS

Signed numbers furnish a convenient means of denoting distance and direction. People have agreed to call distance east of some starting point **positive** and distance west **negative.** Thus, a point may be completely described on a "distance scale" by the use of signed numbers.

E'	D'	C'	B'	A'	S	A	B	C	D	E
-25	-20	-15	-10	-5	0	$+5$	$+10$	$+15$	$+20$	$+25$

Thus, $+10$ denotes 10 units *east* of starting point S; and -20 denotes 20 units *west* of the same starting point.

1. What is the distance from A to E?

2. What is the distance from A' to B?

3. What is the distance from B to C'?

The process of finding the distances in exercises 1, 2, and 3 may be called algebraic subtraction. Notice that the results agree with those of the last section.

Thus, from A to E is 20 units east, or $+25-(+5) = +20$;
from A' to B is 20 units east, or $+10-(-5) = +15$;
from B to C' is 30 units west, or $-20-(+10) = -30$.

Written in columns for subtraction we have:

$$\begin{array}{r} +25 \\ \underline{+\ 5} \\ +20 \end{array} \qquad \begin{array}{r} +10 \\ \underline{-\ 5} \\ +15 \end{array} \qquad \begin{array}{r} -20 \\ \underline{+10} \\ -30 \end{array}$$

Thus, we have a second illustration of subtraction conforming to the first, that is, algebraic subtraction is changed into algebraic addition by changing the sign of the subtrahend.

4. If you should start from zero and travel $+5$ units, then -2 units, where should you be with reference to the starting point?

This illustrates *algebraic addition.* That is, the result is the difference between the absolute values with the sign of the larger prefixed.

Show that the following results conform to the results from algebraic addition:

5. Travel from zero, $+8$, -3, and $+2$. Where should you be?

6. Travel from zero, -2, $+10$, and -3. Where should you be?

7. Travel from zero, -7, $+2$, and $+10$. Where should you be?

8. Travel from zero, -10, -5, and $+12$. Where should you be?

Longitude *east* of the prime meridian is usually indicated by a *plus sign* and longitude *west* by a *minus sign.*

Give the difference in longitude between:

9. A point $+30°$ and one $-10°$.

10. A point $-20°$ and one $-40°$.

11. A point $+10°$ and one $+80°$.

12. The above answers required only the *absolute* number of degrees. In fact, a signed answer could not be given. Why?

13. A man travels from a city, longitude $+3°$, to one whose longitude is $+12°$. Through how many degrees and in what direction did he travel?

14. Show in what way the above solution may be considered algebraic subtraction, and that the answer indicates both the distance (in degree) and the direction.

6. GENERAL OBSERVATION OF SIGNED NUMBERS

You have seen in the general uses of signed numbers that a negative number denotes the opposite of a positive number. If $+$ denotes *gain*, $-$ denotes *loss;* if $+$ denotes direction *east*, $-$ denotes direction *west;* if $+$ denotes temperature *above* zero, $-$ denotes temperature *below* zero.

A negative number may always imply a *reserved subtraction*, so when combined (added) to a positive number it tends to subtract from, or destroy of, the positive number a part equal to itself.

1. If, on two sales, I lose \$15 on one and gain \$25 on the other, what is the net result?

Considered as *algebraic addition* this may be written

$$\begin{array}{r} -\$15 \\ +\$25 \\ \hline +\$10 \end{array}$$
, the result showing a *gain* of \$10.

2. I have \$350 and owe \$500. If I pay the \$350 on my debt, what do I still owe?

As *algebraic addition* this may be written

$$\begin{array}{l} +\$350 \\ \underline{-\$500} \\ -\$150, \text{ showing that I still } \textit{owe } \$150. \end{array}$$

3. Letting + 38 represent the *number and direction* of the automobiles passing a certain point traveling north, and − 48 the *number and direction* of those passing the same point traveling south, indicate that the algebraic sum shows the excess going south.

4. Letting + 2400 represent the number of gallons of water pumped into a tank and −1600 the number drawn out during the same time, add the two numbers and interpret the result.

5. In problem 4, had − 3000 represented the number of gallons withdrawn, what would the algebraic sum indicate?

6. If the lifting force of a balloon is called positive, what must the weight attached be called? Why?

7. If a certain railroad stock sold for 78 yesterday, and to-day for 76, what indicates the change? If it sold for 84 to-day, what indicates the change?

8. If a man loses \$780 on one sale and gains \$1050 on another, what is the net result?

9. If an automobile travels 85 miles directly north and then 110 directly south, express the distance and direction from the starting point.

7. ADDITION AND SUBTRACTION OF LITERAL NUMBERS

In the *formulas* and *equations* used in mathematics it will be necessary to add and subtract algebraic expressions. These follow the principles you have already learned.

NOTE. — A number with no sign is considered positive.

Add:

1.	**2.**	**3.**	**4.**	**5.**	**6.**	**7.**	**8.**
$3a$	$-5b$	$7x$	$-9y$	$6c$	$8e$	$-6m$	$-7n$
$4a$	$3b$	$-3x$	$-2y$	$-7c$	$9e$	$-4m$	$6n$
$-2a$	$-4b$	$-2x$	$8y$	$-5c$	$-7e$	$-3m$	$5n$

9.	**10.**	**11.**	**12.**	**13.**	**14.**	**15.**	**16.**
$2t$	$6z$	$7y$	$-4s$	$-9r$	$-6x$	$7u$	$6t$
$-3t$	$7z$	$-9y$	$-6s$	$3r$	$-3x$	$6u$	$3t$
$5t$	$-5z$	$-5y$	$3s$	$6r$	$-9x$	$5u$	$-4t$
$-4t$	$-2z$	$3y$	$7s$	$-3r$	$8x$	$-9u$	$-9t$

Subtract:

17.	**18.**	**19.**	**20.**	**21.**	**22.**	**23.**	**24.**
$9x$	$-3m$	$6m$	$-9n$	$8s$	$-6s$	$9x$	$-6y$
$6x$	$-9m$	$9m$	$-2n$	$-4s$	$7s$	$-12x$	$-9y$

25.	**26.**	**27.**	**28.**	**29.**	**30.**	**31.**	**32.**
$-14r$	$16t$	$-3r$	$-16t$	$3s$	$16m$	$-21t$	$20x$
$6r$	$-7t$	$-19r$	$8t$	$-15s$	$8m$	$-16t$	$-30x$

Add:

33.	**34.**	**35.**	**36.**	**37.**
$3x+5y$	$-5r-7s$	$-4m-6n$	$5x-9y$	$6x+8y$
$7x-6y$	$-6r+3s$	$10m+7n$	$6x-3y$	$-3x+5y$

38.	39.	40.	41.	42.
$-7t+6n$	$8r-6s$	$9r-3s$	$-6x+7y$	$3m+6n$
$5t-3n$	$5r+3s$	$7r+8s$	$-3x-9y$	$-8m-9n$

Subtract:

43.	44.	45.	46.	47.
$8x-9y$	$9t+3s$	$-3m+2n$	$2r-6s$	$4r-6s$
$3x-4y$	$6t-2s$	$-9m-6n$	$-7r+3s$	$-5r-8s$

48.	49.	50.	51.	52.
$-3x+7y$	$6x-3y$	$2m-9s$	$-3n-2t$	$7x-2y$
$-9x+8y$	$-3x+4y$	$6m-12s$	$-9n+6t$	$2x-8y$

8. REMOVAL OF SIGNS OF GROUPING

When two or more numbers are inclosed in **parentheses,** ( ), they are not to be used singly, but as one number. Thus, in the formula $A=\frac{1}{2}h(b_1+b_2)$, b_1 and b_2 are to be added and used as one number in finding the product. In the formula $S=a-(b+c)$, b and c are to be added and their sum subtracted from a. In removing the parentheses which inclose an expression preceded by a minus sign, it must be observed that the entire expression is to be subtracted and hence that the signs of each term must be changed. This is seen in the following:

$$8-(5-2)=8-3=5.$$

Also $$8-(5-2)=8-5+2=3+2=5.$$

$$10-(3+4)=10-7=3.$$

Also $$10-(3+4)=10-3-4=3.$$

Also $5x-3y-(2x+y)=5x-3y-2x-y=3x-4y.$

Hence we have the principle that

A sign of grouping preceded by the sign — may be removed from an expression if the sign before each term inclosed is changed.

Just as $12+(3+5)=12+8=20,$

or $12+(3+5)=12+3+5=15+5=20,$

so $12x+(3x-x)=12x+2x=14x,$

or $12x+3x-x=15x-x=14x.$

And in general,

A sign of grouping preceded by the sign + may be removed from an expression without changing the sign of any term inclosed.

Remove the sign of grouping and collect:

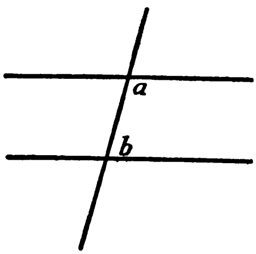
1. $13-(8-2)$.
2. $16-(7+3)$.
3. $14-(6-2)$.
4. $18-(9+3)$.
5. $20+(5+3)$.
6. $4x-(2x-1)$.
7. $6y-(3y+4)$.
8. $3x-y+(x-4y)$.
9. $3x+y-(x+3y)$.
10. $5s-4t-(3s+t)$.
11. $7a+6b-(3a-5b)$.
12. $8r-3s+(r+8s)$.
13. $-3x+6y-(y-8x)$.
14. $7y-3x+(8x+9y)$.
15. Solve $6x-(4x+7)=15$.
16. Solve $8x-(10-2x)=40$.
17. Solve $3x-(2x-3)=8$.
18. Solve $5x+(4-3x)=12$.

19. The sum of two numbers is 20 and their difference is 6. What are the numbers?

SUGGESTION. — Let $n=$ the larger number. Then $20-n=$ the smaller. The equation then becomes $n-(20-n)=6$.

20. The sum of two numbers is 17 and their difference 7. What are the numbers?

21. The angles of a triangle are x, $x + 20°$, and $x - 10°$. Find the number of degrees in each angle.



22. Two parallel lines are cut by a transversal making $\angle b$ 30° less than $\angle a$. Find the size of each angle.

23. If each of the base angles of an isosceles triangle is 15° less than the vertical angle, how many degrees in each?

24. The three angles of a triangle are x, $2x$, and $x - 20°$. Find how many degrees in each.

9. INSERTING SIGNS OF GROUPING

By reversing the process of removing signs of grouping, it follows that

Terms of a polynomial may be inclosed within a sign of grouping when the sign of grouping is preceded by the sign + without changing the sign of the terms inclosed; and when preceded by the sign − by changing the sign of each term inclosed.

Inclose the last three terms within parentheses preceded by the sign +:

1. $a + b - c + d.$
2. $2x + 3y - 2z + 5.$
3. $6x + 2y - 3z - 6.$
4. $5 + n - 2n^2 + n^3.$
5. $x^4 + 3x^3 - 5x^2 + x.$
6. $l + r + 3r^2 + r^3.$

Inclose the last three terms within parentheses preceded by the sign $-$:

7. $7 - x + 3x^2 - x^3$.

8. $8 - 5a - 3b + c$.

9. $a - 2b + 3c + d$.

10. $9x - 3y + 7z + 6$.

11. $5s - 4t + 6n - 7$.

12. $8r - 5s - 3t + 8$.

Inclose the last three terms within a sign of grouping preceded by the sign $+$:

13. $A + B - C - D$.

14. $xy + yz - zw + wx$.

15. $5 - 2n - 9n^2 + n^3$.

16. $r - 4s + 4s^2 + 1$.

17. $m^2 + 6pq - q^2 - 9p^2$.

18. $x^2 + 2xy + y^2 - z^2 + 2zw - w^2$.

19. $16 - 8P + P^2 + 12M - 9M^2 - 4$.

20. $k + 7 + 10k - 1 - 25k^2$.

21. $18 - B^2 - b^2 - Bb$.

22. $1 + 6rr' - r^2 - 9r'^2$.

23–32. Inclose the last three terms in each of the above expressions within a sign of grouping preceded by the sign $-$.

33. In $an - 2t - 3n + bt$, place the terms involving n in parentheses preceded by the sign $+$ and those involving t in parentheses preceded by $-$.

34. In $10P - 2W + mP - nW$, place the terms involving P in parentheses preceded by $+$ and those involving W in parentheses preceded by $-$.

CHAPTER VI

MULTIPLICATION AND DIVISION OF SIGNED NUMBERS

IN the preceding chapter you learned to add and subtract signed numbers. In much of the mathematics that follows it will be necessary to multiply and divide signed numbers by signed numbers. These processes will be shown in this chapter.

1. THE LAW OF SIGNS FOR MULTIPLICATION

In multiplying signed numbers the sign of the product is based upon the following laws:

1. *The product of two factors having like signs is positive.*
2. *The product of two factors having unlike signs is negative.*

The following illustrations will show that the laws are consistent with the meaning we have given signed numbers.

1. If a merchant makes \$1000 per month for 5 months how much better off will he be in five more months?

Evidently the answer is $5 \times \$1000$ or \$5000, as known from the meaning of multiplication in arithmetic.

Calling a profit plus, and time to come plus, this can be written:

$(+5) \times (+\$1000) = +\$5000$, when plus indicates that he is better off.

This illustrates the first law.

2. If a merchant has lost \$1000 per month for the past 5 months, how much better off was he 5 months ago?

Evidently he was \$5000 better off. Using this to illustrate the multiplication of signed numbers, loss must be negative since gain was called positive. And time past must be negative since time to come was called positive. Then we have

$$(-5) \times (-\$1000) = +\$5000.$$

NOTE.—The sign of the product must be +, for it denotes "better off," as in the last problem.

This, too, illustrates the first law.

3. If a merchant is making \$1000 a month, how much worse off was he 5 months ago than now?

Evidently he was \$5000 worse off. Using the signs as in problems 1 and 2 this is,

$$(-5) \times (+\$1000) = -\$5000.$$

Why do you know that the product should be negative?

NOTE.—You cannot give the law of signs as the reason, for this is to illustrate that the law of signs is consistent with facts that we otherwise know.

4. If a merchant is losing \$1000 per month, how much worse off will he be in 5 months?

Evidently he will be \$5000 worse off. Using the signs of the first three problems,

$$(+5) \times (-\$1000) = -\$5000.$$

Problems 3 and 4 illustrate the second law of multiplication.

5. If the temperature starting at zero rises at the rate of 4° per hour for 3 hours, where will it be 3 hours from now?

Evidently it will be 12° above zero.

Thus, $(+3) \times (+4°) = +12°$. What law of multiplication is illustrated?

6. If the temperature now zero has been rising for 3 hours at the rate of 4° per hour, where was it 3 hours ago?

Evidently it was 12° below zero.

Thus, $(-3) \times (+4°) = -12°$. Why was the time called -3? Why was the 4° marked plus? Why was the product marked minus? What law of multiplication is illustrated?

7. If temperature now zero falls at the rate of 4° per hour, where will it be 3 hours from now?

Evidently it will be 12° below zero.

Thus, $(+3) \times (-4°) = -12°$.

Explain the meaning of the signs in the factors and product. What law is illustrated?

8. If temperature now zero has been falling at the rate of 4° per hour, where was it 3 hours ago?

Evidently it was 12° above zero.

Thus, $(-3) \times (-4°) = +12°$.

Explain the meaning of the signs in the factors and product and tell what law is illustrated.

Give the products:

9. $(+3) \times (+5)$.	14. $(+2) \times (-5)$.	19. $(-4) \times (-3)$.
10. $(-3) \times (+5)$.	15. $(+4) \times (-5)$.	20. $(+4) \times (+3)$.
11. $(+3) \times (-5)$.	16. $(-4) \times (-5)$.	21. $(-2) \times (-7)$.
12. $(-3) \times (-5)$.	17. $(-2) \times (-8)$.	22. $(+2) \times (-7)$.
13. $(-2) \times (-4)$.	18. $(+2) \times (-8)$.	23. $(+3) \times (+8)$.

When there are three or more factors, the product is minus or plus depending upon whether the number of minus signs is odd or even.

Give the products:

24. $(+2)(-2)(-2)$.

25. $(-2)(-3)(-1)$.

26. $(+2)(+3)(-1)$.

27. $(+3)(-2)(+2)$.

28. $(+4)(+2)(+1)$.

29. $(-5)(-2)(-2)$.

30. $(+3)(-3)(-3)$.

31. $(-3)(-3)(-3)$.

2. MULTIPLYING LITERAL FACTORS

1. Find the product of $(-3a)(+2b)$.

2. Find the product of $(-3a)(-2b)$.

3. Find the product of $(+2b)(+3c)$.

The order of steps is:

1. *Determine the sign of the product.*

2. *Find the product of the absolute values of the arithmetic factors.*

3. *Find the product of the literal factors.*

Find the products of:

4. $(+5a)(-3b)$.

5. $(-3ab)(-2c)$.

6. $(-4ab)(+2bc)$.

7. $(+3ac)(-2ab)$.

8. $(-2x)(-4xy)$.

9. $(+2a)(-3b)(-2c)$.

10. $(-3x)(-2y)(+4z)$.

11. $(+2a)(-3b)(-2c)$.

12. $(+2ab)(-2bc)(-3ac)$.

13. $(-3d)(-2bd)(-ac)$.

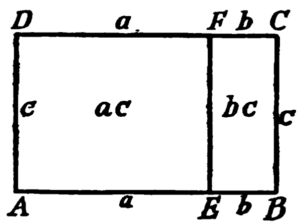
3. MULTIPLYING A POLYNOMIAL BY A MONOMIAL

In simplifying formulas or in solving an equation it is often necessary to multiply a polynomial by a monomial. The process is similar to multiplying arithmetical numbers of two or more orders. Thus, to find 4×36 we really find $4 \times 6 + 4 \times 30$.

So in $3a(2b-4c)$ we have $6ab-12ac$.
That is,

A polynomial is multiplied by a monomial by multiplying each term of the polynomial by the monomial and adding the resulting products.

GEOMETRICAL ILLUSTRATION



Rectangle $ABCD$ is $a+b$ by c. Hence the area is $c(a+b)$. But it is made up of rectangles $AEFD$ and $EBCF$ whose dimensions are respectively a and c, and b and c. Hence their areas are ac and bc. Hence

$$c(a+b)=ac+bc.$$

Find:

1. $3(x-2y)$.
2. $2a(b+3c)$.
3. $2x(3x-2y)$.
4. $7a(3b-4)$.
5. $8c(2c-5)$.
6. $2(6a-7b)$.
7. $3a(2a-5b)$.
8. $3c(a-7b)$.
9. $-5(2a-3c)$.
10. $-2(-4a+2c)$.
11. $-2a(3c-b)$.
12. $-x(3x+2y)$.
13. $-bc(a-3b)$.
14. $4c(-5-6d)$.
15. $3a(2c-5ab)$.

Multiply and unite similar terms:

16. $2(x-3b)+3(2a+b)-4(a-4b)$.

17. $3(2x-y)-4(x+2y)+3(2x-y)$.

18. $3a+2(a+3b)-3(a-2b)+5b$.

19. $4(t+3s)-2(2t-5s)+3(t+2s)+5s$.

20. $2a(b+c)-3a(2b-c)+a(b-5c)+6ac$.

21. $2a(b-2c)-2b(a+c)+3c(2a-3b)-ac$.

22. $x(y+z)-2y(x-3z)+z(2x-5y)+xz$.

4. PRACTICAL USES OF MULTIPLICATION

Many of the problems that arise in mathematics make use of the multiplication you have just had.

Solve:

1. $3(x-7)=6$.
2. $2(3x+1)=32$.
3. $4(2x-5)=28$.
4. $5(3x+4)=50$.
5. $x(x+3)=x^2+15$.
6. $x(2x+1)=2x^2+7$.
7. $x(x-5)=x^2-20$.
8. $x(2x-3)=2x^2-9$.
9. $2x+3(x-6)=17$.
10. $4x-3+2(x+8)=43$.

11. The sum of two numbers is 20. The larger is three times their difference. What are the numbers?

12. Half the larger of the two acute angles of a right triangle is 15° less than the smaller. What are the angles?

SUGGESTION. — Let x = the larger, then $90°-x$ = the smaller. Then $\frac{x}{2}=(90°-x)-15°$.

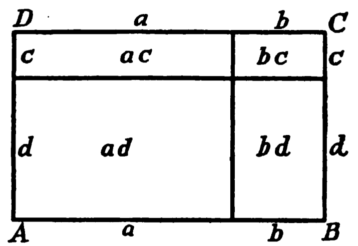
5. MULTIPLICATION OF POLYNOMIALS

Finding the product of two polynomials is like multiplication of two arithmetical numbers of more than one figure each. Thus,

$$24 \times 36 = 4 \times 6 + 4 \times 30 + 20 \times 6 + 20 \times 30.$$

So $(a + b)(c + d) = ac + ad + bc + bd.$

GEOMETRICAL ILLUSTRATION



Rectangle $ABCD$ whose dimensions are $a + b$ and $c + d$ is made up of four rectangles whose dimensions are a and c, b and c, a and d, b and d. So

$$(a + b)(c + d) = ac + ad + bc + bd.$$

Likewise, the product of $2x - 3y$ and $3x + 2y$ is

$$\begin{array}{rl}
2x - 3y & \\
3x + 2y & \\
\hline
6x^2 - 9xy \qquad\quad & [3x(2x - 3y), \text{ 1st partial product.}] \\
4xy - 6y^2 & [2y(2x - 3y), \text{ 2d partial product.}] \\
\hline
6x^2 - 5xy - 6y^2 & [\text{Sum of partial products.}]
\end{array}$$

Thus it is seen that

The product of two polynomials is the sum of all the products obtained by multiplying every term of one polynomial by each term of the other.

Find the products of:

1. $(x+2)(2x+3)$.
2. $(x-3)(2x+1)$.
3. $(2n-3)(n+2)$.
4. $(x-4)(x+3)$.
5. $(2n+3)(3n-2)$.
6. $(x+2y)(2x-y)$.
7. $(3a+b)(2a-3b)$.
8. $(2a-3b)(3a-2b)$.
9. $(2n-3m)(2n+5m)$.
10. $(5n-2m)(2n+m)$.
11. $(4x-3y)(2x+5y)$.
12. $(2m-6)(m-5)$.

We check literal multiplication by assigning arithmetical values to the letters and performing the processes indicated.

Thus find and check the product of

$$(2x-3y)(4x+2y).$$

WORK

$$\begin{array}{l}
2x-3y \\
\underline{4x+2y} \\
8x^2-12xy \\
\underline{\qquad\quad 4xy-6y^2} \\
8x^2-\ \ 8xy-6y^2
\end{array}$$

CHECK

$$\begin{array}{r}
\text{Let } x=2 \text{ and } y=3 \\
4-9=-\ 5 \\
8+6=\ \ \underline{14} \\
\text{Product}=-70 \\
32-48-54=-70
\end{array}$$

NOTE. — Substitute any value for the letters except 1. Substituting 1 checks signs and coefficients only, and not the exponents. Why?

Find and check:

13. $(a-5)(a+7)$.
14. $(2a-8)(3a+2)$.
15. $(2b+c)(b+3c)$.
16. $(2a-3b)(a+4b)$.
17. $(3x+5y)(x+3y)$.
18. $(x-7y)(3x+2y)$.
19. $(m-3n)(2m+3n)$.
20. $(3t-2u)(2t+5u)$.
21. $(2r+5s)(r-6s)$.
22. $(r-8t)(r+2t)$.
23. $(4d-e)(d+6e)$.
24. $(2a+b)(a-3c)$.
25. $(3a-b)(2c+4d)$.
26. $(a-3b)(2c-5d)$.

27. Find $3\frac{3}{4} \times 5\frac{2}{3}$ as the product of two polynomials.

WORK

$$\begin{array}{l} 3 + \frac{3}{4} \\ \underline{5 + \frac{2}{3}} \\ 15 + \frac{15}{4} + \frac{6}{3} + \frac{6}{12} = 15 + 3\frac{3}{4} + 2 + \frac{1}{2} = 21\frac{1}{4}. \end{array}$$

Find:

28. $2\frac{1}{2} \times 3\frac{3}{4}$.

29. $3\frac{1}{3} \times 5\frac{1}{2}$.

30. $4\frac{2}{3} \times 2\frac{1}{2}$.

31. $5\frac{1}{2} \times 4\frac{3}{4}$.

32. $6\frac{2}{3} \times 3\frac{1}{4}$.

33. $5\frac{1}{4} \times 3\frac{1}{3}$.

34. $3\frac{1}{4} \times 5\frac{1}{2}$.

35. $6\frac{2}{3} \times 4\frac{1}{4}$.

36. $7\frac{1}{2} \times 6\frac{2}{3}$.

37. Find $7\frac{1}{2} \times 7\frac{1}{2}$.

$$\begin{array}{l} \quad 7 + \frac{1}{2} \\ \quad \underline{7 + \frac{1}{2}} \\ 49 + 7 \times \frac{1}{2} \\ \qquad \underline{7 \times \frac{1}{2} + \frac{1}{4}} \\ 49 + 7 + \frac{1}{4} = 56 + \frac{1}{4} \end{array}$$

38. Find $5\frac{1}{2} \times 5\frac{1}{2}$.

$$\begin{array}{l} \quad 5 + \frac{1}{2} \\ \quad \underline{5 + \frac{1}{2}} \\ 25 + 5 \times \frac{1}{2} \\ \qquad \underline{5 \times \frac{1}{2} + \frac{1}{4}} \\ 25 + 5 + \frac{1}{4} = 30 + \frac{1}{4} \end{array}$$

39. Find $(a + \frac{1}{2})(a + \frac{1}{2})$.

$$\begin{array}{l} a + \frac{1}{2} \\ \underline{a + \frac{1}{2}} \\ a^2 + \frac{1}{2}a \\ \qquad \underline{\frac{1}{2}a + \frac{1}{4}} \\ a^2 + a + \frac{1}{4} = a(a+1) + \frac{1}{4} \end{array}$$

Exercises 37, 38, and 39 show an important use of algebraic numbers. Exercises 37 and 38 were the squares of mixed numbers whose fractional parts were one-half. Exercise 39, by using the general number a for the whole number, represented *any* such numbers. So its product can be interpreted as the form of *any* such square. By factoring $a^2 + a$ into $a(a + 1)$ the result becomes a formula easily used for such squares. Thus, $(4\frac{1}{2})^2 = 20\frac{1}{4}$ (*i.e.* $4 \times 5 + \frac{1}{4}$).

At sight give:

40. $2\frac{1}{2} \times 2\frac{1}{2}$.	**44.** $3\frac{1}{2} \times 3\frac{1}{2}$.	**48.** $1\frac{1}{2} \times 1\frac{1}{2}$.
41. $4\frac{1}{2} \times 4\frac{1}{2}$.	**45.** $8\frac{1}{2} \times 8\frac{1}{2}$.	**49.** $10\frac{1}{2} \times 10\frac{1}{2}$.
42. $5\frac{1}{2} \times 5\frac{1}{2}$.	**46.** $7\frac{1}{2} \times 7\frac{1}{2}$.	**50.** $11\frac{1}{2} \times 11\frac{1}{2}$.
43. $6\frac{1}{2} \times 6\frac{1}{2}$.	**47.** $9\frac{1}{2} \times 9\frac{1}{2}$.	**51.** $12\frac{1}{2} \times 12\frac{1}{2}$.

52. Find $8\frac{1}{2} \times 4\frac{1}{2}$.

$$\begin{array}{l} 8 + \frac{1}{2} \\ \underline{4 + \frac{1}{2}} \\ 32 + 4 \times \frac{1}{2} \\ \underline{\quad\quad 8 \times \frac{1}{2} + \frac{1}{4}} \\ 32 + 12 \times \frac{1}{2} + \frac{1}{4} = 38\frac{1}{4} \end{array}$$

53. Find $3\frac{1}{2} \times 7\frac{1}{2}$.

$$\begin{array}{l} 3 + \frac{1}{2} \\ \underline{7 + \frac{1}{2}} \\ 21 + 7 \times \frac{1}{2} \\ \underline{\quad\quad 3 \times \frac{1}{2} + \frac{1}{4}} \\ 21 + 10 \times \frac{1}{2} + \frac{1}{4} = 26\frac{1}{4} \end{array}$$

54. Find $(a + \frac{1}{2})(b + \frac{1}{2})$.

$$\begin{array}{l} a + \frac{1}{2} \\ \underline{b + \frac{1}{2}} \\ ab + \frac{1}{2}b + \frac{1}{2}a + \frac{1}{4} = ab + \frac{1}{2}(b + a) + \frac{1}{4} \end{array}$$

This, again, is an important formula for the product of *any* two mixed numbers whose fractions are each $\frac{1}{2}$.

The product translated into a rule is,

To find the product of any two mixed numbers whose fractions are each $\frac{1}{2}$, to the product of the whole numbers add half their sum and $\frac{1}{4}$.

Observe that when the sum of the whole number is even, the product ends in $\frac{1}{4}$. When odd, it ends in $\frac{3}{4}$. Thus, $3\frac{1}{2} \times 7\frac{1}{2} = 26\frac{1}{4}$, and $4\frac{1}{2} \times 9\frac{1}{2} = 42\frac{3}{4}$.

At sight give:

55. $2\frac{1}{2} \times 6\frac{1}{2}$.	**59.** $5\frac{1}{2} \times 7\frac{1}{2}$.	**63.** $2\frac{1}{2} \times 7\frac{1}{2}$.
56. $2\frac{1}{2} \times 9\frac{1}{2}$.	**60.** $3\frac{1}{2} \times 9\frac{1}{2}$.	**64.** $3\frac{1}{2} \times 4\frac{1}{2}$.
57. $3\frac{1}{2} \times 5\frac{1}{2}$.	**61.** $4\frac{1}{2} \times 8\frac{1}{2}$.	**65.** $3\frac{1}{2} \times 6\frac{1}{2}$.
58. $4\frac{1}{2} \times 6\frac{1}{2}$.	**62.** $2\frac{1}{2} \times 5\frac{1}{2}$.	**66.** $4\frac{1}{2} \times 5\frac{1}{2}$.

67. $4\frac{1}{2} \times 7\frac{1}{2}$.	**70.** $3\frac{1}{2} \times 11\frac{1}{2}$.	**73.** $5\frac{1}{2} \times 9\frac{1}{2}$.
68. $3\frac{1}{2} \times 8\frac{1}{2}$.	**71.** $4\frac{1}{2} \times 10\frac{1}{2}$.	**74.** $6\frac{1}{2} \times 8\frac{1}{2}$.
69. $2\frac{1}{2} \times 10\frac{1}{2}$.	**72.** $5\frac{1}{2} \times 8\frac{1}{2}$.	**75.** $6\frac{1}{2} \times 10\frac{1}{2}$.

6. HOW LITERAL MULTIPLICATION IS USED IN MAKING FORMULAS

You have seen in the preceding exercises how literal notation enabled you to make formulas that shortened work. Another interesting example is given here.

You have already learned that when two successive commercial discounts are quoted, one is first taken off and then the second is taken on the former net price.

Thus, \$500 less 20% and 10% means that 20% of \$500, or \$100, is first deducted and then 10% of the remaining \$400, or \$40, is deducted.

This may be done as follows:

$$\begin{array}{ll} \$500, & \text{list price.} \\ \underline{.80} & (100\% - 20\%). \\ \$400 & \\ \underline{.90} & (100\% - 10\%) \\ \$360, & \text{net cost.} \end{array}$$

Using literal notation the process may be expressed by the formula

$$\textbf{Net cost} = (100\% - r\%)(100\% - r_1\%)(\textbf{list price}).$$

Finding the product of $100\% - r\%$ and $100 - r_1\%$, the work is as follows:

$$\begin{array}{l} 100\% - r\% \\ \underline{100\% - r_1\%} \\ 100\% - r\% \quad \text{(for 100\% of anything is the thing itself)} \\ \underline{\qquad\quad - r_1\% + r_1\% \times r\%} \\ 100\% - r\% - r_1\% + r_1\% \times r\% \end{array}$$

Hence

$$\text{Net cost} = (100\,\% - r\,\% - r_1\,\% + r_1\,\% \times r\,\%) \times \text{list price},$$
$$= [100\,\% - (r\,\% + r_1\,\% - r_1\,\% \times r\,\%)] \times \text{list price}.$$

Hence successive discounts of $r\,\%$ and $r_1\,\%$ are equal to the single discount of $(r\,\% + r_1\,\% - r_1\,\% \times r\,\%)$, for being inclosed in parentheses, this is considered a single number.

Translated into a rule this is,

To find the single discount equal to two successive discounts, subtract their product from their sum.

Thus, $700 less 30 % and 20 % is the same as $700 less 44 % (30 % + 20 % − 6 %).

At sight give the single discount equal to:

1. 10 % and 10 %.
2. 20 % and 10 %.
3. 40 % and 10 %.
4. 40 % and 25 %.
5. 50 % and 10 %.
6. 20 % and 20 %.
7. 20 % and 25 %.
8. 40 % and 25 %.
9. 60 % and 20 %.
10. 60 % and 10 %.
11. 60 % and 25 %.
12. 30 % and 10 %.
13. 40 % and 15 %.
14. 50 % and 40 %.
15. 40 % and 30 %.
16. 30 % and 30 %.
17. 50 % and 20 %.
18. 50 % and 30 %.
19. 50 % and 50 %.
20. 40 % and 40 %.

7. SPECIAL PRODUCTS

Exercises in the last section illustrated the advantages of finding formulas for special products. There are certain **special products** so useful in mathematics that they should be remembered, just as the *multiplication table* in arithmetic was remembered.

1. Find $(a+b)(a+b)$ or $(a+b)^2$.

WORK

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad\;\; ab+b^2} \\ a^2+2\,ab+b^2 \end{array}$$

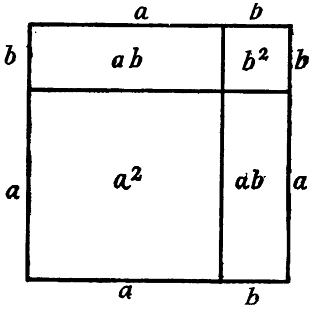
This product becomes a formula for the square of the sum of any two numbers whether literal or arithmetical numbers. Translated into words it is,

The square of the sum of two numbers is the square of the first, plus twice their product, plus the square of the second.

Or, as a formula, it is

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

GEOMETRICAL ILLUSTRATION



The square whose side is $a+b$ is made up of two squares and two rectangles as shown in the figure. So the large square whose area is $(a+b)^2$ is composed of figures whose areas are a^2, ab, ab, and b^2. That is, $(a+b)^2 = a^2 + 2\,ab + b^2$.

At sight give the squares:

1. $(a+3)^2$.	**6.** $(2x+3y)^2$.	**11.** $(5+m)^2$.
2. $(b+2)^2$.	**7.** $(4s+3t)^2$.	**12.** $(5+2n)^2$.
3. $(2a+b)^2$.	**8.** $(5s+2t)^2$.	**13.** $(5x+3y)^2$.
4. $(a+3b)^2$.	**9.** $(m+5n)^2$.	**14.** $(7+3s)^2$.
5. $(2x+3)^2$.	**10.** $(2m+7n)^2$.	**15.** $(7s+4t)^2$.

16. Use the formula to square 14.

WORK

$14^2=(10+4^2)=100+80+16=196.$

Find by the formula:

17. 13^2.	**20.** 17^2.	**23.** 24^2.	**26.** 32^2.
18. 15^2.	**21.** 18^2.	**24.** 25^2.	**27.** 35^2.
19. 16^2.	**22.** 19^2.	**25.** 28^2.	**28.** 42^2.

29. Use the formula to square $7\frac{1}{2}$.

WORK

$(7\frac{1}{2})^2=(7+\frac{1}{2})^2=49+7+\frac{1}{4}=56\frac{1}{4}.$

Find by the formula:

30. $(5\frac{1}{2})^2$.	**32.** $(9\frac{1}{2})^2$.	**34.** $(6\frac{1}{3})^2$.	**36.** $(9\frac{1}{3})^2$.
31. $(8\frac{1}{2})^2$.	**33.** $(8\frac{1}{4})^2$.	**35.** $(4\frac{1}{4})^2$.	**37.** $(8\frac{3}{4})^2$.

38. Derive a formula for $(a-b)^2$.

WORK

$$\begin{array}{l}
a - \quad b \\
\underline{a - \quad b} \\
a^2 - \quad ab \\
\underline{\quad\;\; - \quad ab + b^2} \\
a^2 - 2ab + b^2
\end{array}$$

State in words the fact expressed by the product.

Memorize.

$$(a-b)^2 = a^2 - 2ab + b^2.$$

Find by the formula:

39. $(x - 2y)^2$.
40. $(x - 5)^2$.
41. $(y - 8)^2$.
42. $(2a - 3)^2$.
43. $(3a - 2b)^2$.
44. $(2c - 8)^2$.
45. $(3c - 2d)^2$.
46. $(c - 7d)^2$.
47. $(5 - 3x)^2$.
48. $(6 - 2y)^2$.
49. $(3x - 5y)^2$.
50. $(2a - 5b)^2$.

51. By the formula find the square of 49.

WORK

$$49^2 = (50 - 1)^2 = 2500 - 100 + 1 = 2401.$$

Find by the formula:

52. 29^2.
53. 28^2.
54. 38^2.
55. 39^2.
56. 49^2.
57. 48^2.
58. 69^2.
59. 89^2.
60. 58^2.
61. 78^2.
62. 68^2.
63. 87^2.

64. Make a formula for $(a - b)(a + b)$.

If correct, you found

$$(a - b)(a + b) = a^2 - b^2.$$

Stated in words, this formula is,

The product of the sum and the difference of two numbers is the difference of their squares.

By the formula find:

65. $(x - 2)(x + 2)$.
66. $(2 - y)(2 + y)$.
67. $(x - 2y)(x + 2y)$.
68. $(x - 5)(x + 5)$.
69. $(2x - 3y)(2x + 3y)$.
70. $(3a - b)(3a + b)$.

71. $(2a - 3b)(2a + 3b)$.

72. $(4 - 3a)(4 + 3a)$.

73. $(3x + 5y)(3x - 5y)$.

74. $(4x - 7y)(4x + 7y)$.

75. $(5t - 8s)(5t + 8s)$.

76. $(6n - 5m)(6n + 5m)$.

77. Find by the formula 48×52.

WORK

$$48 \times 52 = (50 - 2)(50 + 2) = 2500 - 4 = 2496.$$

Find by the formula:

78. 38×42.

79. 41×39.

80. 62×58.

81. 63×57.

82. 49×51.

83. 88×92.

84. 82×78.

85. 75×65.

86. 45×55.

87. 75×85.

88. 68×72.

89. 95×85.

SUMMARY

You have now had three of the most important *special products* of algebra. They are grouped here for reference.

$$\text{I.}\quad (a + b)^2 = a^2 + 2ab + b^2.$$

$$\text{II.}\quad (a - b)^2 = a^2 - 2ab + b^2.$$

$$\text{III.}\quad (a + b)(a - b) = a^2 - b^2.$$

8. TWO OTHER SPECIAL PRODUCTS

You have seen the value of the three formulas found in the last section and how they save work in finding special products of literal or arithmetical numbers. There are *two* other important special products taken up in this section.

1. Find the product of $(x + 2)(x + 3)$.
2. Find the product of $(x - 2)(x - 3)$.
3. Find the product of $(x + 2)(x - 3)$.
4. Find the product of $(x - 2)(x + 3)$.

If you examine the four products just found you will see that

1. *The first term of each product is the square of* x *(the common term);*

2. *The second term of each is* x *(the common term) with a coefficient that is the algebraic sum of the other two terms;*

3. *The last term of each is either* $+6$ *or* -6, *the product of the other two terms.*

5. Write expressions for two binomials having a common term.

6. Will $x+a$ and $x+b$ answer question 5? Why?

7. Find the product of $(x+a)(x+b)$.

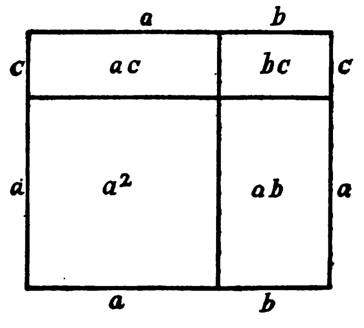
If correct, you have

$$(x+a)(x+b)=x^2+(a+b)x+ab.$$

Stated in words this is,

The product of two binomials having a common term is the square of the common term, plus the algebraic sum of the other two terms as the coefficient of the common term, plus the product of the other two terms.

GEOMETRICAL ILLUSTRATION



Thus, we see that the area $(a+b)(a+c)$ is composed of $a^2+ab+ac+bc$ or $a^2+(b+c)a+bc$.

By inspection give:

8. $(x+4)(x+2)$.
9. $(x+2)(x+5)$.
10. $(x+3)(x+5)$.
11. $(x+4)(x+5)$.
12. $(x-2)(x-4)$.
13. $(x-3)(x-5)$.
14. $(x-4)(x-5)$.
15. $(x-2)(x-10)$.
16. $(a-3)(a+4)$.
17. $(a-5)(a+2)$.
18. $(a-6)(a+2)$.
19. $(b+3)(b-4)$.
20. $(b+4)(b-5)$.
21. $(x+2)(x-8)$.
22. $(x-3)(x-10)$.
23. $(x+4)(x+6)$.
24. By the formula find 23×28.

$23 \times 28 = (20+3)(20+8) = 400+220+24 = 644$.

In the same way give:

25. $(10+3)(10+7)$.
26. $(10+2)(10+8)$.
27. $(10+7)(10+5)$.
28. $(20+1)(20+5)$.
29. $(20+4)(20+6)$.
30. 24×26.
31. 37×33.
32. 46×44.
33. 52×58.
34. 64×66.
35. 81×89.
36. 72×78.
37. 67×63.
38. 84×86.
39. 79×71.

9. PRODUCTS OF TWO BINOMIALS HAVING CORRESPONDING SIMILAR TERMS

1. Find the product of $(2x+1)(3x+4)$.
2. Find the product of $(2x-1)(3x-4)$.
3. Find the product of $(2x+1)(3x-4)$.
4. Find the product of $(2x-1)(3x+4)$.

If you study the products found you see that

1. *The first and last terms of each product are the products of the corresponding similar terms.*

2. *The middle term of each product is the algebraic sum of the products of the dissimilar terms.*

The formula is therefore

$$(ax + b)(cx + d) = acx^2 + (bc + ad)x + bd.$$

ILLUSTRATIVE EXAMPLE

$$(3a + 2)(2a - 5)$$

LONG WAY

$$\begin{array}{r}
3a + 2 \\
2a - 5 \\
\hline
6a^2 + 4a \\
-15a - 10 \\
\hline
6a^2 - 11a - 10
\end{array}$$

SHORT WAY

$$\overbrace{(3a + 2)\ (2a - 5)}^{-15a} = 6a^2 - 11a - 10$$
$$\underbrace{2)\ (2a}_{+4a}$$

NOTE. — In using the "short way" all work can be done mentally and the product written down directly from the factors. The illustration is to show the order of thinking out the cross products to be added.

Find by the cross-product method:

1. $(a + 3)(2a + 5)$.	11. $(2a + b)(a + 3b)$.
2. $(2a - 3)(a - 5)$.	12. $(3a - b)(2a + 3b)$.
3. $(3a - 1)(2a + 3)$.	13. $(2x - y)(5x + 2y)$.
4. $(2x - 5)(3x + 1)$.	14. $(3x - 2y)(5x - y)$.
5. $(x + 7)(2x - 3)$.	15. $(4a + c)(a + 3c)$.
6. $(c + 4)(3c - 1)$.	16. $(5a - 2c)(2a + 5c)$.
7. $(3c + 2)(2c + 3)$.	17. $(5a + 2c)(2a - 5c)$.
8. $(3c - 2)(2c - 3)$.	18. $(2x - y)(4x - y)$.
9. $(3c - 2)(4c - 1)$.	19. $(4y - a)(3y - 5a)$.
10. $(2b + 5)(2b + 3)$.	20. $(4x - 3y)(2x - 7y)$.

10. DIVISION OF SIGNED NUMBERS

You know from arithmetic that *division* is the process of finding one of two numbers when the product and one number are known ; you also know that

Quotient × Divisor = Dividend

From this we may infer the **law of signs in division.**

Since $(+2)\times(+3)=+6$, $(+6)\div(+2)=+3$.
Since $(-2)\times(+3)=-6$, $(-6)\div(-2)=+3$.
Since $(-2)\times(-3)=+6$, $(+6)\div(-2)=-3$.
Since $(+2)\times(-3)=-6$, $(-6)\div(+2)=-3$.

Hence we see that

1. *If the dividend and divisor have like signs, the quotient is positive.*

2. *If the dividend and divisor have unlike signs, the quotient is negative.*

Give the quotient at sight:

1. $(+18)\div(+3)$.
2. $(-18)\div(-3)$.
3. $(+18)\div(-3)$.
4. $(-18)\div(+3)$.
5. $(-5)\div(-5)$.
6. $(-5)\div(+5)$.
7. $(+5)\div(-5)$.
8. $(-8)\div(+1)$.
9. $(-8)\div(-1)$.
10. $(+4\,ab)\div(+2\,a)$.
11. $(-4\,ab)\div(-2\,a)$.
12. $(+15\,x)\div(-3\,x)$.
13. $(+6\,a)\div(-1)$.
14. $(-6\,a)\div(-1)$.
15. $(+9\,ab)\div(-3\,a)$.
16. $(-9\,ab)\div(-3\,a)$.
17. $(+16\,x)\div(-8\,x)$.
18. $(-16\,x)\div(-8\,x)$.
19. $(+24\,xy)\div -6\,x)$.
20. $(-24\,xy)\div(+6\,xy)$.
21. $(-12\,ac)\div(-6\,ac)$.
22. $(+12\,ac)\div(-3\,a)$.

23. Give the quotient at sight: $(-15y) \div (+5y)$.

24. Give the quotient at sight: $(-14x) \div (-2x)$.

It is more common to express a division in algebra as a fraction. Thus, $(+3ab) \div (-2a)$ is expressed $\frac{+3ab}{-2a}$. In algebra, as in arithmetic,

Both terms of a fraction (or a division) may be divided by the same factor without changing the value of the fraction (quotient).

Thus, $\frac{+36ab}{-4ac} = -\frac{9b}{c}$.

The expression $\frac{a}{b}$ may be read "a divided by b" or, as more commonly, "a over b."

Simplify:

25. $\frac{2 \times 4}{-4x}$.

26. $\frac{-6ab}{3b}$.

27. $\frac{-18xy}{-12x}$.

28. $\frac{12ac}{-4b}$.

29. $\frac{16 \times 7}{-12y}$.

30. $\frac{18abc}{10ac}$.

31. $\frac{24}{-9y}$.

32. $\frac{36}{-18ab}$.

33. $\frac{54z}{36xz}$.

11. THE LAW OF EXPONENTS IN DIVISION

Since $a^5 \div a^2 = \frac{a^5}{a^2} = \frac{aa\cancel{aaa}}{\cancel{aa}} = a^3$, and

$$b^7 \div b^4 = \frac{bbb\cancel{bbbb}}{\cancel{bbbb}} = b^3, \text{ so in general,}$$

In dividing powers of the same base, the exponent of the quotient is obtained by subtracting the exponent of the divisor from the exponent of the dividend.

As a formula, this is

$$a^m \div a^n = a^{m-n}.$$

Thus, $(6\,x^4y^6) \div (-2\,xy^4) = -3\,x^3y^2$,
and $(-24\,x^3y^6) \div (-4\,x^3y) = 6\,y^5$.

NOTE. — The x^3 of the dividend canceled the x^3 of the divisor, hence no x appeared in the quotient.

Divide:

1. $a^{10} \div a^4$.
2. $x^6 \div x^2$.
3. $s^9 \div s^7$.
4. $n^{12} \div n^8$.
5. $4\,b^2 \div 2\,b$.
6. $8\,c^4 \div 4\,c^2$.
7. $a^4b^2 \div a^2b$.
8. $a^6b^2 \div a^2b^2$.
9. $6\,a^4b \div 3\,ab$.
10. $10\,x^{10}y^2 \div 2\,x^8y$.
11. $16\,a^5b^3 \div 4\,a^3b^3$.
12. $18\,a^7c \div 9\,a^3c$.
13. $9a^{10} \div -3\,a^6$.
14. $10\,a^5b^2 \div -2\,ab$.
15. $-8\,x^6y^2 \div 2\,x^2y$.
16. $4\,x^5y^5 \div -2\,x^3y^2$.
17. $-8\,x^6y \div 2\,x^4$.
18. $-24\,a^3bc \div 6\,a^2b$.
19. $-30\,abc \div -5\,a$.
20. $28\,a^3bc \div -7\,abc$.
21. $-6\,a^2b^2c \div a^2b^2$.
22. $42\,x^3y \div 6\,xy$.
23. $-56\,abc \div 8\,ac$.
24. $-21\,m^2n \div 7\,n$.
25. $38\,r^3s \div -2\,r^3$.
26. $-63\,n^3t \div 7\,nt$.
27. $-24\,a^2bc \div -8\,a^2$.
28. $35\,xyz \div -5\,x$.
29. $100\,x^2yz \div 50\,xyz$.
30. $-50\,m^6n^2 \div 25\,m^5n$.
31. $16\,a^3bc \div -a^2bc$.
32. $27\,x^5y^2 \div -3\,xy^2$.

12. DIVIDING A POLYNOMIAL BY A MONOMIAL

Since division is the inverse of multiplication, *quotient* × *divisor* = *dividend.* That is, the product of each term of the quotient by the divisor must give the dividend. Hence the rule,

To divide a polynomial by a monomial, divide each of its terms by the monomial.

Thus, $$\frac{4a^3 - 12a^2b^3}{2a^2} = 2a - 6b^3.$$

And $$\frac{x^{10} - 5x^4 + 3x^2}{-x^2} = -x^8 + 5x^2 - 3.$$

Give the quotients:

1. $\frac{81ab - 18ac}{3a}$.
2. $\frac{30x^3 - 40x^2y}{-5x^2}$.
3. $\frac{2ac - 4ab + 6ad}{2a}$.
4. $\frac{-6x^2y^3 + 18x^3y^2 - 12x^2y}{6x^2y}$.
5. $\frac{4a^4x - 3a^3x^2 + 2a^2x^3}{-ax}$.
6. $\frac{x^2y^2 - xy + 4x^3y}{xy}$.
7. $\frac{x^{10} - 5x^4 + 3x^3}{-x^3}$.
8. $\frac{3a^5 - 6a^4b + 9a^3b^2}{3a^3}$.

Divide:

9. $x^3 - x^2$ by $-x^2$.
10. $27x^9 - 18x^7$ by $9x^5$.
11. $-a + b - c$ by -1.
12. $4x^4y^6 + 8x^7y^5$ by $4x^3y^3$.
13. $6x^3y^2 - 9x^4y$ by $3xy$.
14. $a^4 + 3ab - 6a^2b^2$ by a^2.
15. $4n^4 - 6n^2m$ by $-2n^2$.
16. $12m^2n^2 - 18m^4n^6$ by $6m^2n$.

13. DIVIDING A POLYNOMIAL BY ANOTHER

The work of dividing one polynomial by another is shown by the following example:

1. Divide $x^3 + x^4 + 7x^2 - 6x + 8$ by $x^2 + 2x + 8$.

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x^2 + 2x + 8 & x^4 + x^3 + 7x^2 - 6x + 8 \\
 & \underline{x^4 + 2x^3 + 8x^2} \\
 & - x^3 - x^2 - 6x \\
 & \underline{- x^3 - 2x^2 - 8x} \\
 & x^2 + 2x + 8 \\
 & \underline{x^2 + 2x + 8}
\end{array}
$$

Observe that the dividend and the divisor were arranged according to descending powers of x.

Study the work and arrangement as you carefully read the explanation.

The product of the term of highest power in x in the quotient and the term of highest power in the divisor must give the term of highest power in the dividend. Hence the highest term of the quotient is obtained by dividing the highest term of the dividend, x^4, by the highest term of the divisor, x^2. This gives x^2, the first term of the quotient.

Multiply the whole divisor by the term of the quotient just found. This gives $x^4 + 2x^3 + 8x^2$, which is placed below the dividend.

The dividend is the product of the divisor by the whole quotient. Hence subtracting the product, $x^4 + 2x^3 + 8x^2$, from the dividend, the remainder, $-x^3 - x^2 - 6x + 8$, must be the product of the divisor by the part of the quotient to be found.

Therefore, the product of the next highest term of the quotient by the highest term of the divisor must equal the highest term of the remainder. Hence dividing $-x^3$ of the remainder by x^2 of the divisor gives $-x$, the second term of the quotient.

Multiply the whole divisor by the new term, $-x$; subtract the product from the remainder. This leaves x^2+2x+8.

Evidently the third term of the quotient will be obtained from this second remainder just as the second term was obtained from the first remainder.

By continuing this process, all of the terms of the quotient may be found.

If in any problem the divisor is an exact divisor of the dividend, the work may be carried on until a remainder zero is found. Otherwise the work may be continued until a remainder is obtained in which the highest term of the remainder is of lower power than the highest term in the divisor. This is a true remainder.

Hence the rule for dividing one polynomial by another:

1. *Arrange both the dividend and the divisor according to the descending or ascending powers of some letter, the same letter being used in both.*

2. *Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.*

3. *Multiply the whole divisor by this term of the quotient, and subtract the result from the dividend.*

4. *Treat the remainder as a new dividend (having the terms arranged as before), and repeat the process, continuing until either the remainder zero, or a true remainder, is found.*

NOTE. — In the exercises only binomial divisors are used.

EXAMPLE 1. Divide $a^2 - 11a + 30$ by $a - 5$.

$$\begin{array}{r|l} & a - 6 \\ \hline a - 5 & a^2 - 11a + 30 \\ & \underline{a^2 - 5a} \\ & -6a + 30 \\ & \underline{-6a + 30} \end{array}$$

CHECK.—When $a=8$, dividend $= 6$, divisor $= -2$, and quotient $= -3$, as it should.

EXAMPLE 2. Divide $x^4 + y^4$ by $x + y$.

$$\begin{array}{r|l} & x^3 - x^2y + xy^2 - y^3 \\ \hline x + y & x^4 \qquad\qquad\qquad + y^4 \\ & \underline{x^4 + x^3y} \\ & -x^3y \qquad\qquad + y^4 \\ & \underline{-x^3y - x^2y^2} \\ & x^2y^2 \quad + y^4 \\ & \underline{x^2y^2 + xy^3} \\ & -xy^3 + y^4 \\ & \underline{-xy^3 - y^4} \\ & 2y^4 \end{array}$$

In this example there is a true remainder $2y^4$ obtained. By using the fractional form to indicate the division of the remainder, as in arithmetic, the entire quotient in this problem may be expressed thus:

$$\frac{x^4 + y^4}{x + y} = x^3 - x^2y + xy^2 - y^3 + \frac{2y^4}{x + y}.$$

CHECK.—When $x = 2$ and $y = 3$, dividend $= 97$, divisor $= 5$, quotient $= 19\frac{2}{5}$, as it should.

NOTE.—Observe that the terms in the dividend, divisor, and all the products and remainders in the above example are arranged in descending powers of x. Similar terms should be kept in columns in the work.

Divide and check:

1. $a^2 + 3a + 2$ by $a + 1$.
2. $X^2 + 3X - 40$ by $X + 8$.
3. $x^2 - 5x + 6$ by $x - 2$.
4. $P^2 + 7P + 12$ by $P + 3$.
5. $t^2 - 6t + 8$ by $t - 4$.
6. $y^2 - 9y + 20$ by $y - 4$.
7. $H^2 - H - 30$ by $H - 8$.
8. $b^2 + b - 20$ by $b + 5$.
9. $m^2 + 3m - 40$ by $m - 10$.
10. $2x^2 + 3x + 1$ by $2x + 1$.
11. $3x^2 + 16x - 12$ by $x + 6$.
12. $12K^2 - 14K - 6$ by $4K - 6$.
13. $6V^2 + 29V + 35$ by $3V + 7$.
14. $4A^2 - 9$ by $2A + 3$.
15. $25R^2 - 4$ by $5R - 2$.
16. $4p^2 - 1$ by $2p + 1$.
17. $1 - 9a^2$ by $1 - 3a$.
18. $16 - 25D^2$ by $4 + 5D$.
19. $a^2 - b^2$ by $a - b$.
20. $x^2 - 4y^2$ by $x + 2y$.
21. $W^2 - 25g^2$ by $W - 5g$.
22. $m^2 + 5mn + 4n^2$ by $m + 4n$.
23. $P^2 - 9PQ + 20Q^2$ by $P - 5Q$.
24. $x^2 + 21 + 10x$ by $3 + x$.
25. $1 + 2a + a^2$ by $a + 1$.
26. $15y + 36 + y^2$ by $y + 4$.
27. $12E^3 - 30E^2 + 2E - 5$ by $2E - 5$.
28. $2r^3 - 3r^2 + 3 + 4r$ by $3 + 2r$.
29. $B^6 - 1$ by $B - 1$.
30. $y^3 - z^3$ by $y + z$.
31. $W^4 - V^4$ by $W^2 + V^2$.
32. $H^6 - t^6$ by $H^3 + t^3$.
33. $144R^2 - 1$ by $12R + 1$.

14. HOW SIGNED NUMBERS SIMPLIFY THE STEPS IN THE SOLUTION OF AN EQUATION

In solving equations, you often find it necessary to add some number to both sides of the equation or to subtract a number from both sides of it. The work may be shortened now that we understand how to add or subtract signed numbers. This is shown in the following problems.

1. Solve $x - 5 = 4$.

Adding $+5$ to both members,

$$x - 5 + 5 = 4 + 5.$$

Or $$x = 9.$$

Since 5 was added to $x - 5$ to cancel the -5 and give but x in the left-hand member, we need not write down $x - 5 + 5$, but could have written $x = 4 + 5$ from the given equation.

Here we observe that the resulting equation may be considered as made up from the given equation by "changing the sign of -5 to $+5$" and **transposing** it to the other side of the equation.

2. Solve $x + 3 = 8$.

Subtracting 3 from both members,

$$x + 3 - 3 = 8 - 3,$$

Or $$x = 8 - 3.$$

This again gives an equation which is the same as if the $+3$ had been changed to -3 and *transposed* to the other side.

3. Solve $3x - 5 = 2x + 3$.

Adding $+5$ and subtracting $2x$ from both members,

$$3x - 2x = 3 + 5.$$

Here again we have a resulting equation in which $+2x$ and -5 were each transposed to opposite sides of the equation, with their signs changed.

And in general,

Any term in an equation may be transposed from one side of the equation to the other by changing its sign.

4. Solve $5x - 20 = 4x + 5$.

Transposing -20 and $+4x$ gives

$$5x - 4x = 5 + 20.$$

Collecting terms, $x = 25$.

5. Solve $3(x+1) = 12 + 4(x-1)$.

Removing parentheses, $3x + 3 = 12 + 4x - 4$.

Transposing, $3x - 4x = 12 - 4 - 3$.

Collecting, $-x = 5$.

Dividing by -1, $x = -5$.

Checking, $3(-5+1) = 12 + 4(-5-1)$

$-12 = -12$.

Solve by transposing terms, and check:

1. $3x - 6 = 2x + 1$.
2. $2x - 7 = 5x + 5$.
3. $4x - 8 = 9 - 2x$.
4. $2x + 4 = 5x - 12$.
5. $x - 3 = 4x - 9$.
6. $x - (6 - 2x) = 9(x - 1)$.
7. $(x-2)(x-3) = (x-4)(x-5)$.
8. $3(x+1) = 5(1-x)$.
9. $2x - 6 = 5(20 - x)$.
10. $2x - 3(x-4) = 2x - 3$.
11. $(5-x)(1+x) = (2-x)(4+x)$.
12. $(x-1)(x-2) = (x+3)(x-4)$.
13. $(x-4)(x+4) = x^2 - 8x$.
14. $n - 6 - 9(n-1) = -2n$.
15. $2n - 3(n+1) = 14$.
16. $(3n+7)(1+4n) = 12n^2 + 39$.
17. $2x + 2(x-1) = 3x - 7$.
18. $5(2x+1) = 3(2x-7) + 58$.
19. $(n-1)(n-2) = n^2 - n - 12$.
20. $3x - 4 = 5(x-1) + 3$.

CHAPTER VII

ALGEBRAIC FACTORS AND FRACTIONS

In algebra, as in arithmetic, it is often convenient, in performing multiplications and divisions, to recognize the **factors** of an expression; that is, the expressions that multiplied together made the given expression. Just as in arithmetic, 3 and 7 are the factors of 21 because $3 \times 7 = 21$, so in algebra the factors of $a^2 - b^2$ are $a - b$ and $a + b$, for $(a - b)(a + b) = a^2 - b^2$. It is only by first knowing the products that we can factor such numbers as 25, 18, 24, 36, etc.; so in algebra, factoring comes from the knowledge you have of multiplication.

1. HOW TO TAKE OUT A MONOMIAL FACTOR

You learned in multiplication that $m(a + b) = ma + mb$. So the factors of $ma + mb$ are m and $a + b$. That is,

$$\boldsymbol{ma + mb = m(a + b).}$$

The steps are:

1. *Find, by inspection, the factor common to all the terms.*

2. *Divide each term of the polynomial by this common monomial factor.*

3. *The divisor and quotient are the monomial and polynomial factors, respectively, of the given polynomial.*

4. *Check the work by multiplying the two factors together to see if the product equals the given expression.*

Factor by inspection and check by multiplication:

1. $2a+2b$.	5. $5a^2+10b^2$.	9. $4bx+12by$.
2. $5x-5y$.	6. $4a^2-8bc$.	10. $3x^2+6x$.
3. $3x+6y$.	7. $2ab+2ac$.	11. $3a+3b-6c$.
4. $4a-8b$.	8. $3ax-6ay$.	12. $5ab-5ac+5ax$.

13. $2x^2+4xy+8y^2$.	17. $3x^2+9x-15$.
14. $a^2x+ax-a^2y$.	18. $6a^3-9a^2+12a$.
15. $2ax-4a^2xy+2a^3y$.	19. $2\pi r^2+2\pi r+2\pi$.
16. $3a^3b-6a^2b^2+9ab^3$.	20. $a^2x^2+4a^2x^2y^2-6axy$.

2. HOW TO FACTOR THE DIFFERENCE OF TWO SQUARES

Since you learned under "special products" that $(a+b)(a-b)=a^2-b^2$, you know that the factors of a^2-b^2 are $(a+b)$ and $(a-b)$. That is,

Since $$(a+b)(a-b)=a^2-b^2,$$

Then $$a^2-b^2=(a+b)(a-b).$$

Observe that

1. *This type form has but two terms, and that they are connected by a minus sign.*

2. *Each term must be the square of some expression.*

3. *The factors are each binomials whose terms are alike, but the terms of one factor are connected by a plus sign and of the other by a minus sign.*

4. *The terms of the factors when squared give the corresponding terms of the given expression.*

1. Factor $4x^2 - 9$.

$4x^2$ is the square of $2x$, and 9 is the square of 3, and the squares are connected with a minus sign. Hence

$$4x^2 - 9 = (2x+3)(2x-3).$$

CHECK. — $(2x+3)(2x-3) = 4x^2 - 9$.

Factor by inspection and check by multiplication:

2. $n^2 - 4$.
3. $a^2 - 9$.
4. $y^2 - 16$.
5. $1 - x^2$.
6. $9 - y^2$.
7. $x^2 - 25$.
8. $a^2 - 4b^2$.
9. $x^2 - 9y^2$.
10. $a^2 - 25$.
11. $9 - 4b^2$.
12. $a^2 - 16b^2$.
13. $m^2 - 49$.
14. $x^2 - 36y^2$.
15. $4a^2 - 25b^2$.
16. $9x^2 - 4y^2$.
17. $16a^2 - 9b^2$.
18. $x^2 - 64y^2$.
19. $16x^2 - y^2$.
20. $x^2 - 49y^2$.
21. $4x^2 - 9y^2$.
22. $36a^2 - 25b^2$.
23. $2x^2 - 2y^2$.
(Take out the monomial factor and factor the resulting factor.)
24. $3a^2 - 12b^2$.
25. $5m^2 - 80n^2$.
26. $6x^2 - 24y^2$.

3. HOW TO FACTOR TRINOMIALS THAT ARE THE SQUARES OF BINOMIALS

You learned under "special products" that $(a+b)^2 = a^2 + 2ab + b^2$, and that $(a-b)^2 = a^2 - 2ab + b^2$. Hence you know that the factors of $a^2 + 2ab + b^2$ are $(a+b)$ and $(a+b)$, and of $a^2 - 2ab + b^2$ are $(a-b)$ and $(a-b)$. That is,

Since $(a+b)^2 = a^2 + 2ab + b^2$, **and** $(a-b)^2 = a^2 - 2ab + b^2$.

Then $a^2 + 2ab + b^2 = (a+b)(a+b)$.

And $a^2 - 2ab + b^2 = (a-b)(a-b)$.

Observe that

1. *This type is a trinomial, two of whose terms are positive terms that are the squares of two numbers.*

2. *The third term (usually written between the other two) is twice the product of the numbers of which the other two terms are the squares* (i.e. *twice the product of the square roots of the other terms*).

3. *This third term may be either positive or negative and determines the sign between the two terms of the factors.*

1. Factor $x^2 + 6x + 9$.

x^2 is the square of x, and 9 is the square of 3. The product of 3 and x is $3x$. $6x$ is twice $3x$. Hence

$$x^2 + 6x + 9 = (x+3)(x+3)$$

CHECK. — $(x + 3)(x + 3) = x^2 + 6x + 9$.

Factor by inspection and check by multiplication:

2. $a^2 + 8a + 16$.
3. $a^2 - 8a + 16$.
4. $x^2 + 6xy + 9y^2$.
5. $x^2 - 6xy + 9y^2$.
6. $4a^2 + 12ab + 9b^2$.
7. $4a^2 - 20ab + 25b^2$.
8. $x^2 + 10xy + 25y^2$.
9. $9x^2 - 24x + 16$.
10. $y^2 + 14y + 49$.
11. $4x^2 - 4xy + y^2$.
12. $9n^2 - 6mn + m^2$.
13. $x^2 - 16x + 64$.
14. $9a^2 + 30ab + 25b^2$.
15. $x^2 + y^2 - 2xy$.
16. $a^2 + b^2 + 2ab$.
17. $25 + a^2 - 10a$.
18. $a^2 + 36 + 12a$.
19. $x^2 + 1 - 2x$.

4. TRINOMIALS COMPOSED OF TWO BINOMIALS HAVING A COMMON TERM

You learned that $(x+2)(x+3) = x^2 + 5x + 6$, or more generally $(x+a)(x+b) = x^2 + (a+b)x + ab$. Hence

$$\boldsymbol{x^2 + (a+b)x + ab = (x+a)(x+b)}.$$

1. Factor $x^2 + 5x + 4$.

Observing this type, we see that we are to find two numbers whose product is 4 and whose sum is 5. By inspection and trial we find they are 4 and 1, for $1 \times 4 = 4$ and $1 + 4 = 5$. Hence

$$x^2 + 5x + 4 = (x+1)(x+4).$$

CHECK. — $(x+1)(x+4) = x^2 + 5x + 4$.

2. Factor $x^2 - 5x + 4$.

By inspection we see that the sum of -1 and -4 is -5 and that the product of $(-1)(-4)$ is 4. Hence

$$x^2 - 5x + 4 = (x-4)(x-1).$$

CHECK. — $(x-4)(x-1) = x^2 - 5x + 4$.

3. Factor $y^2 + 3y - 18$.

We are to find two numbers whose product is -18, and whose sum is $+3$. Hence one factor must be positive and one negative, and the larger one must be positive. Evidently they are $+6$ and -3, for $(-3)(+6) = -18$, and $+6 - 3 = +3$. Hence

$$y^2 + 3y - 18 = (y-3)(y+6).$$

CHECK. — $(y-3)(y+6) = y^2 + 3y - 18$.

Factor by inspection and check by multiplication:

4. $x^2 + 8x + 12$.
5. $x^2 - 8x + 12$.
6. $x^2 + 9x + 14$.
7. $a^2 + 9a + 20$.
8. $a^2 - 10a + 9$.
9. $a^2 - 10a + 21$.
10. $y^2 + 11y + 24$.
11. $y^2 + 11y + 30$.
12. $x^2 - 12x + 20$.
13. $c^2 + 13c + 30$.
14. $b^2 + 12b + 32$.
15. $b^2 - 5b + 6$.
16. $n^2 - 9n + 14$.
17. $z^2 - 9z + 20$.
18. $t^2 - 10t + 9$.
19. $r^2 - 10r + 16$.
20. $x^2 - 8x + 7$.
21. $x^2 + 2x - 3$.
22. $a^2 + 3a - 4$.
23. $b^2 + 3b - 18$.
24. $x^2 + 3x - 28$.
25. $c^2 - 4c - 12$.
26. $a^2 - 5a - 14$.
27. $c^2 - 4c - 21$.
28. $x^2 + 7x + 12$.
29. $r^2 - 5r + 6$.
30. $y^2 - 8y + 7$.

5. TRINOMIALS COMPOSED OF ANY TWO BINOMIALS

You learned under "special products" that $(ax+b)(cx+d)=acx^2+(bc+ad)x+bd$. Hence

$$acx^2+(bc+ad)x+bd=(ax+b)(cx+d)$$

Studying the multiplication below, we may see how to factor this type.

$$\begin{array}{c} ax+b \\ \times \\ cx+d \\ \hline acx^2+(bc+ad)x+bd \end{array}$$

Observe that

1. *The first and third terms of the given trinomial are the products of the first terms and second terms, respectively, of the factors.*

2. *The coefficient of x, the only common term, is the sum of the cross products ad and bc.*

1. Factor $3x^2+11x+10$.

We are seeking two factors of $3x^2$ and of $+10$ whose cross products give $+11x$. The method is a "cut and try" method. That is, it consists in guessing the correct pair from the possible pairs. There are 4 possible pairs shown below.

1	*2*	*3*	*4*
$3x+2$	$x+2$	$3x+1$	$x+1$
$x+5$	$3x+5$	$x+10$	$3x+10$

By inspection, the second pair is the only one whose cross products give a sum of $+11x$. Hence

$$3x^2+11x+10=(x+2)(3x+5).$$

CHECK. — $(x+2)(3x+5)=3x^2+11x+10.$

2. Factor $4a^2 - 13a + 10$.

Since $+10$ is the product of two factors with *like signs*, and $-13a$ is the sum of the cross products, the sign of the second term of each factor must be *minus*. We try successively the following possible pairs (there are six in all):

1	*2*	*3*	*4*
$2a-5$	$2a-10$	$4a-2$	$4a-5$
$2a-2$	$2a-1$	$a-5$	$a-2$

At once we see that the first three pairs are each impossible, for one factor of each pair has a factor of 2, which is not a factor of the given expression. Trying the fourth pair, the sum of the cross products is $-13a$. Hence

$$4a^2 - 13a + 10 = (4a-5)(a-2).$$

CHECK. — $(4a-5)(a-2) = 4a^2 - 13a + 10$.

Factor and check by multiplication:

3. $10x^2 + 17x + 7$.

4. $3x^2 + 10x + 3$.

5. $5a^2 + 12a + 4$.

6. $2b^2 + 9b + 10$.

7. $3a^2 + 10a + 7$.

8. $8x^2 + 22x + 15$.

9. $3a^2 - 11a + 10$.

10. $10c^2 - 17c + 7$.

11. $3x^2 - 10x + 3$.

12. $5a^2 - 12a + 4$.

13. $8b^2 - 22b + 15$.

14. $2y^2 - 9y + 10$.

15. $2c^2 - 11c + 5$.

16. $3t^2 - 8t + 4$.

17. $4r^2 + 3r - 27$.

18. $6x^2 + 17x - 14$.

19. $3a^2 + 2a - 5$.

20. $3x^2 + x - 4$.

21. $3b^2 + b - 2$.

22. $4y^2 - y - 5$.

23. $3c^2 - 2c - 5$.

24. $2x^2 + x - 28$.

6. A SUMMARY OF FACTORING METHODS

1. *Look for a common monomial factor. If one is discovered, divide it out.*

2. *If the resulting factor is a binomial, see if it is the difference of two squares. If so, factor it.*

3. *If the resulting factor is a trinomial, see if it is of the form* $\boldsymbol{a^2 \pm 2\,ab + b^2}$ *or* $\boldsymbol{x^2 + (a+b)x + ab}$. *If so, factor it.*

4. *If not of the above forms, use the "cut and try" method to see if it can be factored.*

5. *Always check the result by finding the product of the factors.*

Factor by any method you can, and check:

1. $a^3 + 3\,a^2 + a$.	11. $r^3 - 2\,r^2 - 15\,r$.
2. $4\,x^2 + 12\,xy + 9\,y^2$.	12. $an^2 + 13\,an + 42\,a$.
3. $m^2n + 6\,mn + 9\,n$.	13. $a^2 + a - 42$.
4. $a^2 + 8\,a + 12$.	14. $2\,a^2 + 3\,a + 1$.
5. $at^2 - 8\,at + 12\,a$.	15. $2\,xy^2 + 5\,xy + 2\,x$.
6. $y^2 + 4\,y - 12$.	16. $t^2 + 11\,t - 26$.
7. $an^2 - 4\,an - 12\,a$.	17. $6\,a^2 + 11\,a + 3$.
8. $a^3 + 8\,a^2 + 15\,a$.	18. $x^2 - 16\,x + 64$.
9. $t^3 + 2\,t^2 - 15\,t$.	19. $ay^2 + 8\,ay + 16\,a$.
10. $ax^2 - 4\,ay^2$.	20. $x^2 + 2\,xy - 35\,y^2$.

7. THE MEANING OF A FRACTION

You learned in arithmetic that division could be indicated by writing the dividend over the divisor. Thus,

$$7 \div 3 = \tfrac{7}{3} = 2\tfrac{1}{3};\quad 9 \div 4 = \tfrac{9}{4} = 2\tfrac{1}{4};\quad 3 \div 7 = \tfrac{3}{7};\quad \text{etc.}$$

In algebra the sign of division is seldom used, but the division is indicated by writing the dividend over the divisor.

Thus, $m \div n$ is usually written $\frac{m}{n}$, and $(a+b) \div (a-b)$ is written $\frac{a+b}{a-b}$. These indicated divisions are called **fractions.** They are read "m divided by n" and "$a+b$ divided by $a-b$," or more briefly "m over n" and "$a+b$ over $a-b$." The same rules that apply to arithmetical fractions apply to algebraic fractions.

8. THE SIGNS OF A FRACTION

In an algebraic fraction there are three signs to consider:

(1) *The sign before the fraction;* (2) *the sign of the numerator; and* (3) *the sign of the denominator.*

Since an algebraic fraction is an indicated division, it must follow the *law of signs* of division. Thus,

$$\frac{+a}{+b} = +\frac{a}{b}; \quad \frac{-a}{-b} = +\frac{a}{b}; \quad \frac{-a}{+b} = -\frac{a}{b}; \quad \frac{+a}{-b} = -\frac{a}{b}.$$

It follows from this, that to preserve the algebraic value of a fraction

1. *If the signs of both terms of a fraction are changed, the sign before the fraction must be left unchanged.*

2. *If the sign of but one term of a fraction is changed, the sign before the fraction must be changed.*

Write with positive numerators and denominators, placing the proper sign before the fraction:

1. $\frac{-a}{b}$.	4. $\frac{-2a}{3b^2}$.	7. $\frac{9x}{-11y}$.	10. $\frac{-(a+b)}{(a-b)}$.
2. $\frac{x}{-y}$.	5. $\frac{6xy}{-5z}$.	8. $\frac{-at}{-sm}$.	11. $\frac{-x(y+z)}{+y(x+z)}$.
3. $\frac{-x}{-y}$.	6. $\frac{-7n}{-12m^2}$.	9. $\frac{-3}{5xy}$.	12. $\frac{a(c+b)}{-b(a-c)}$.

9. REDUCING FRACTIONS TO LOWER TERMS

Just as like factors may be canceled from both terms of the fractions of arithmetic, so they may be canceled from both terms of an algebraic fraction without changing the value of the fraction. Thus, just as $\frac{12}{20}=\frac{3}{5}$ by canceling 4 from each term, so $\frac{xy}{yz}=\frac{x}{z}$ by canceling y from each term.

Reduce to lowest terms:

1. $\frac{10}{12}$.
2. $\frac{9}{12}$.
3. $\frac{15}{18}$.
4. $\frac{21}{28}$.
5. $\frac{12}{20}$.
6. $\frac{36}{96}$.
7. $\frac{42}{54}$.
8. $\frac{72}{100}$.
9. $\frac{240}{800}$.
10. $\frac{60}{720}$.
11. $\dfrac{ab}{ac}$.
12. $\dfrac{3x}{6x^2}$.
13. $\dfrac{4m^2n}{6mn^2}$.
14. $\dfrac{12p^4q^2}{16p^3q^5}$.
15. $\dfrac{4abc}{6a^2bd}$.
16. $\dfrac{5xyz^2}{10yz}$.
17. $\dfrac{3w^2v^3}{12wv^4}$.
18. $\dfrac{7rst^2}{21r^2st}$.
19. $\dfrac{8Mv^2}{36v^5}$.
20. $\dfrac{4R^2}{R^3}$.
21. $\dfrac{4R(R+H)}{8R^2(R+H)}$.
22. $\dfrac{a^2-b^2}{a^2+3ab+b^2}$.
23. $\dfrac{xn-xm}{n^2-m^2}$.
24. $\dfrac{A^3-AB^2}{A^2+2AB+B^2}$.
25. $\dfrac{R^2-1}{2Ry+2y}$.
26. $\dfrac{t^2+2t-15}{t^2-t-6}$.
27. $\dfrac{1-p^2}{1-2p+p^2}$.
28. $\dfrac{4n^2+4n+1}{4n^2-1}$.
29. $\dfrac{v^2-w^2}{(v+w)^2}$.

10. FRACTIONS REDUCED TO MIXED EXPRESSIONS

You have learned in arithmetic that when the numerator is as great or greater than the denominator, it may be divided by the denominator and its value thus given as a whole number or as a whole number and a fraction. So in an algebraic fraction, which is only an indicated division, the numerator

may be divided by the denominator when the degree of the numerator in any letter is as great as or greater than that of the denominator. Thus, just as $\frac{7}{4} = 1\frac{3}{4}$, so $\frac{3a^2}{a} = 3a$, and $\frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}$, found by division.

Reduce to mixed expressions:

1. $\frac{5}{3}$.
2. $\frac{12}{7}$.
3. $\frac{13}{4}$.
4. $\frac{15}{8}$.
5. $\frac{p^2+1}{p-1}$.
6. $\frac{4x^2+1}{2x-1}$.
7. $\frac{x^2+y^2}{x+y}$.
8. $\frac{x^3-y^3}{x+y}$.
9. $\frac{n^2-5n+1}{n-2}$.
10. $\frac{1-x^3}{1+x-x^2}$.
11. $\frac{x^3+16}{x+2}$.
12. $\frac{y^3+8}{y^2-y+2}$.

11. ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

Just as $\frac{1}{8} + \frac{3}{8} = \frac{1+3}{8}$, or $\frac{4}{8}$, so $\frac{a}{x} + \frac{b}{x} = \frac{a+b}{x}$.

And just as

$$\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{8+9}{12}, \text{ so } \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

That is,

To add (or subtract) algebraic fractions, change them to a common denominator, find the sum (or difference) of the numerators, and place the result over the common denominator.

Add:

1. $\frac{3a}{4} + \frac{5a}{8}$.
2. $\frac{a}{b} + \frac{a}{2b}$.
3. $\frac{ab}{c} + \frac{a}{d}$.

4. $\frac{3x}{2a}+\frac{x}{5}$.
5. $\frac{4x}{7}+\frac{x-2}{5}$.
6. $\frac{1}{xy}+\frac{1}{xz}$.
7. $\frac{a}{bc}+\frac{b}{ac}$.
8. $\frac{x}{x+y}+\frac{y}{x-y}$.
9. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

12. MULTIPLYING ALGEBRAIC FRACTIONS

Just as $\frac{2}{3}\times\frac{5}{7}=\frac{10}{21}$, and $\frac{\cancel{4}}{5}\times\frac{3}{\underset{2}{\cancel{8}}}=\frac{3}{10}$,

So $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, and $\frac{\cancel{x}y}{z}\times\frac{a}{b\cancel{x}}=\frac{ay}{bz}$.

Multiply:

1. $\frac{x}{y}\times\frac{a}{m}$.
2. $\frac{4am}{5b}\times\frac{c}{d}$.
3. $\frac{5m}{n}\times\frac{a}{m}$.
4. $\frac{bm^2}{ac}\times\frac{c}{d}$.
5. $\frac{4xy}{6}\times\frac{3ab}{4y}$.
6. $\frac{2ax}{by}\times\frac{by}{ax}$.
7. $\frac{a^2b}{cd}\times\frac{ac^2}{b^2d}$.
8. $\frac{x^4y^2}{a^3}\times\frac{a^2x}{by^2}$.
9. $\frac{x^2}{y^2}\times\frac{y^2z}{x^3}$.
10. $\frac{a^4}{b^2c}\times\frac{ab^2}{a^2}$.
11. $\frac{3a}{x+y}\times\frac{c}{x+y}$.
12. $\frac{a+b}{a-b}\times\frac{a^2-b^2}{a+b}$.

13. DIVISION OF ALGEBRAIC FRACTIONS

In arithmetic you learned to divide one fraction by another by inverting the divisor and multiplying. In the same way,

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

Divide:

1. $\frac{a}{2}\div\frac{3m}{4a}$.
2. $\frac{3ay}{4n}\div\frac{3am}{2n}$.
3. $\frac{4a^2}{9m}\div\frac{2a}{3}$.
4. $\frac{8a}{9m}\div\frac{a}{n}$.
5. $\frac{3x^2y}{5m^2}\div\frac{2xy^2}{10m}$.
6. $\frac{20x^2}{21a^2}\div\frac{4ax}{3a^2}$.

7. $\frac{5\,xy}{4\,z} + \frac{3\,y^2}{10\,z^2}$.

8. $\frac{5\,m^2}{n} + \frac{4\,mn}{7\,a^2}$.

9. $\frac{a+b}{x} + \frac{a-b}{y}$.

10. $\frac{a+1}{a} + \frac{a^2-1}{a^2}$.

11. $\frac{2\,b^2}{a^2-b^2} + \frac{b}{a+b}$.

12. $\frac{6a+2}{4-a^2} + \frac{a^2}{2+a}$.

14. FRACTIONAL EQUATIONS

In some problems of mathematics, the equations are stated in the form of a fraction. How to solve such equations is shown here.

1. Solve $\frac{3}{2\,x} = \frac{1}{4}$.

SOLUTION

Observe that $4\,x$ will contain both denominators. Hence
Multiplying both terms by $4\,x$,

$$6 = x,$$

Or $$x = 6, \text{ the solution.}$$

CHECK. $$\frac{3}{2 \times 6} = \frac{1}{4}.$$

2. Solve $\frac{x+1}{x} = \frac{1}{x} - \frac{2}{3\,x}$.

Multiplying both members by $3\,x$,

$$3(x+1) = 3 - 2.$$

Removing parentheses, and collecting,

$$3\,x + 3 = 1.$$

Subtracting 3 from both members,

$$3\,x = -2.$$

Dividing by 3,

$$x = -\tfrac{2}{3}.$$

Check the result.

Solve and check:

3. $\frac{4}{x} = \frac{5}{x} - 1$.

4. $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 1$.

5. $\frac{4}{5x} - \frac{7}{10x} = \frac{1}{10}$.

6. $\frac{45}{x} + 7 = 12$.

7. $\frac{1+x}{2} = 1$.

8. $\frac{1}{x} = \frac{2}{x-1}$.

9. $\frac{x+3}{x-3} = 3$.

10. $\frac{x-3}{x+9} = \frac{x-5}{x+5}$.

11. $\frac{9}{5x+2} = \frac{7}{3x+4}$.

12. $\frac{x}{x-2} + \frac{6}{x+2} = 1$.

13. If a fireman has enough coal to fire one furnace 8 days, or a smaller one 10 days, how long will it last if both are fired?

SOLUTION

Let x = number of days the coal will last both. Then $\frac{1}{x}$ represents the part of it that is burned per day. Also the first furnace uses $\frac{1}{8}$ of it per day, and the second $\frac{1}{10}$ of it. Hence the two use $\frac{1}{8} + \frac{1}{10}$ of it per day. Hence the equation

$$\frac{1}{8} + \frac{1}{10} = \frac{1}{x}.$$

Solve the equation as in exercises 3–12.

14. One boy can mow a lawn in 3 hours and another can mow it in 4 hours. How long at this rate will it take both?

15. It takes one pump twice as long to fill a reservoir as it does another. If the two together can fill it in 4 hours, how long will it take each working alone?

16. A tank is fitted with two pipes, one of which can empty it in 30 min. and the other in 20 min. If the tank is $\frac{3}{4}$ full, how long will it take to empty it if both pipes are left open?

17. A board 12 ft. long is to be cut into two pieces so that the longer length will contain 3 of the shorter length. Find the length of each.

CHAPTER VIII

INDIRECT MEASUREMENT

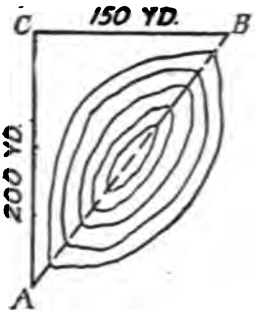
1. FINDING HEIGHTS AND DISTANCES: SCALE DRAWINGS

You have seen lines used to represent the relative lengths of various objects. In the bar graphs that you have already studied, you saw various magnitudes represented by the length of lines. You are also familiar with the maps of geography and have noticed a "scale" on these maps from which you could measure and compute the distance between any two places. In addition to these, you may have seen shop drawings and blue prints used by engineers and architects.

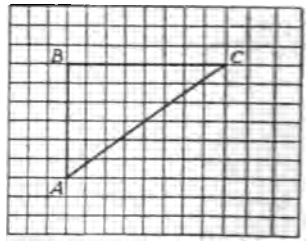
All these maps and plans are "scale drawings" in which real lengths are represented by shorter lines whose relations to the real lengths are known.

The ratio of any two lines in these scale drawings is the same as the ratio of the real lengths which they represent.

When distances between two objects cannot be measured directly on account of some intervening object, as a house, a swamp, or a lake, other measurements that can be made are taken and a "drawing to a scale" is made, from which the required distance can be computed. Squared paper is very convenient to use in drawing a figure to scale.

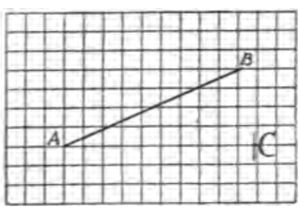


1. A man wanted to know the distance between two points A and B. On account of a small lake lying between the two, direct measurement was impossible. So he measured direct north from A 200 yards to C, a point directly west of B, and then direct east 150 yards to B. Using a scale 1 in. = 40 yards, make a scale drawing and find the distance.



2. Use squared paper in making the scale drawing of problem 1 and see if your two results agree.

3. If the length of each square in the diagram represents 10 ft., how long a line is represented by AB? By BC? By AC?



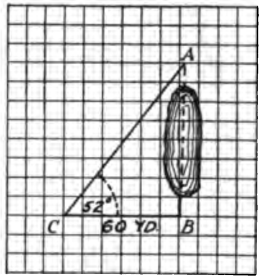
4. Using a pair of compasses, measure the length of AB (in squares) by placing one point at A and adjusting the compasses so the other point will fall at B. Now describe an arc about A as center cutting either line upon which A lies and count the squares. Measure AC in problem 3 in this way.

5. If a man walks 100 yards due north and then 80 yards due east, find by a scale drawing how far he is from the starting place.

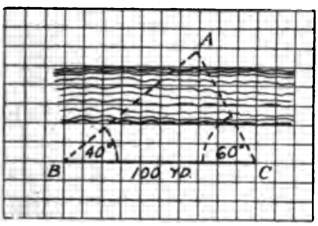
6. Two men start from the same point. One walks 200 yards east then 100 yards south. The other walks 150 yards north and then 80 yards west. How far apart are they?

7. A baseball diamond is 90 feet square. By a scale drawing, find how far it is from the first to the third base.

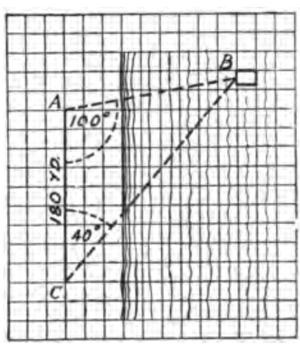
8. To find the distance from a point at B to a point at A which he could not measure directly, a man measured 60 yards to a point C and then 100 yards to point A, noting that the line CA made an angle of 52 degrees with CB. Make a scale drawing and find the distance.



9. To find the distance from point B to a point A on the opposite side of a river, a surveyor measured to a point C 100 yards away and on the same side of the river. From B he found angle $CBA = 40°$, and from C, angle $BCA = 60°$. By a scale drawing, find the distance from B to A.

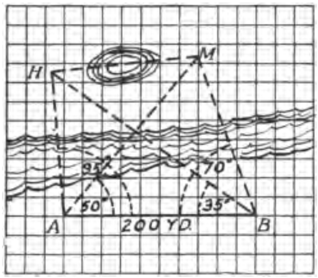


10. A boy swam from a boat house at A to a diving dock at B and wished to know the distance. From a point C he observed $\angle ACB = 40°$. Then he measured to A and found it to be 180 yards, and that the angle $CAB = 100°$. Make a scale drawing and find how far he swam.



11. A man at A wishes to know the distance from a house at H to a mill at M, both on opposite sides of a river from him and on opposite sides of a lake from

each other. From A he sights a tree at B and notes that $\angle BAH = 95°$ and that $\angle BAM = 50°$. He then measures to B and finds the distance to be 200 yards; and from B he



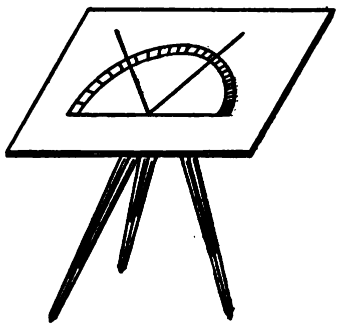
finds that $\angle ABH = 35°$ and that $\angle ABM = 70°$. Make a scale drawing and compute the distance from H to M.

2. ACTUAL PROBLEMS IN INDIRECT MEASUREMENT

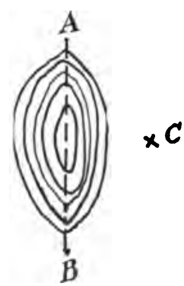
It will be both interesting and instructive to use scale drawing to find inaccessible distances by actual measurements. The engineer uses *steel tape* or a *surveyor's chain* for measuring distances, and a *transit* for measuring angles. But a pupil can easily devise crude instruments at little if any cost that are sufficiently accurate for his purpose, which is merely to gain a knowledge of principles used. A yardstick or a tape measure can be used to measure distance; and a large protractor mounted on a drawing board and with movable arms hinged at the center may be used for measuring the angles.

A TRANSIT

1. If you wish to find the distance from a point to some inaccessible point, what information must you have before constructing the scale drawing?

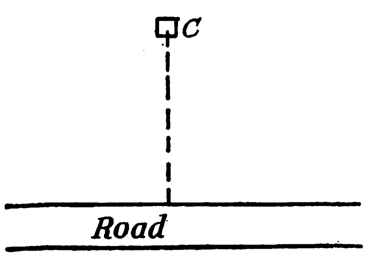


2. Show what two measures you would need to find the distance from A to B. What other measure is necessary?



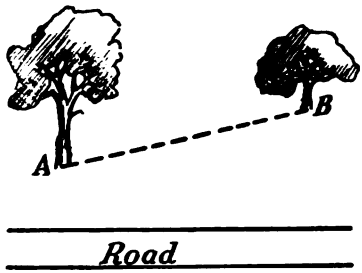
3. Show how you could find the distance from A to B by measuring one length measure only from C.

4. Measure the distance between two accessible points by the two methods suggested in problems 2 and 3. Check your results by actual measurements. By which method were you more accurate?



5. Suppose that C is a club house and that you wish to know the shortest distance from the club house to the road. Show what measurements are necessary.

6. If you can find a corresponding problem in real measurements, make the necessary measurements and compute the distance.

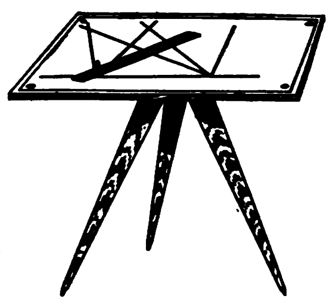


7. Suppose that A and B are two trees in a field. Show what measurements you can take along the road in order to make a scale drawing and compute the distance from one to the other.

8. Find a similar problem to which you can apply real measurements and thus find the distance.

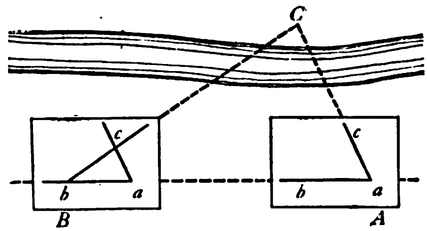
3. USING A PLANE TABLE TO FIND DISTANCE

An instrument called a **plane table** is easily constructed and can be used in finding distances, through scale drawing, and in making a map of a lake, pond, river bank, or any small area. It consists of a drawing board mounted on a tripod as shown in the figure in the margin.



In using the plane table for finding distances, as for example the distance from A to an inaccessible point C, a piece of paper is fastened on the board and a pin stuck through the paper into the board at a point directly over A. A straightedge is placed against the pin and pointed toward a second point B which is accessible, and line ab is drawn. Then the straightedge is pointed to C and line ac is drawn. The plane table is now taken to point B and b is

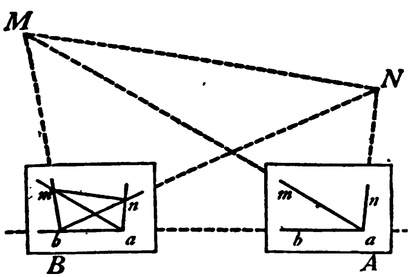
placed directly over B, ba falling along line BA. The straightedge is then sighted to C and a line bc is drawn cutting ac in c. The figure abc is an exact representation of ABC.



1. Suppose AB is 200 yards, and ab, on the paper, is 10 inches, what is the scale upon which triangle abc, representing ABC, is drawn?

2. If the scale is 1 in. = 20 yards, and $ac = \frac{3}{4}$ inch, how far is it from A to C?

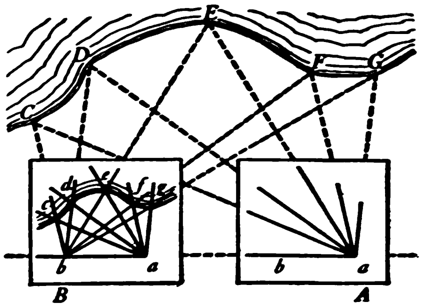
3. Supposing the distance between two points M and N is wanted, proceed as follows: Select two accessible points A and B. Place the plane table directly over A so that a pin at a will be directly over point A. Draw ab, am, and an toward points B, M, N. Now place the plane table over B, and then make lines bn and bm intersecting an and am as shown in the figure. If $AB = 400$ ft. and $ab = 8$ in., what is the scale of figure $abmn$?



4. If scale $abmn$ is 1 in. = 50 ft. and mn is 12 in., how far is it from M to N?

5. By use of a plane table, find by indirect measurement the distance from a point to some other point. Check your work by direct measurement to see how accurate you were.

6. By use of a plane table, find by indirect measurement the distance between two points. Check your work by direct measurement to see how accurate you were.

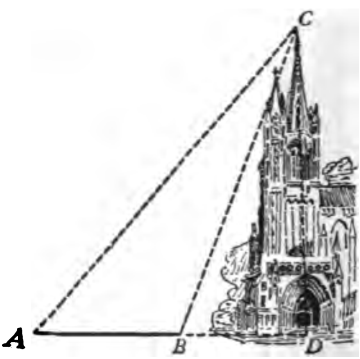


7. The following drawing shows the method of making a map $cdefg$ of a river bank $CDEFG$ by using a plane table. Study the drawing and explain how points c, d, e, f, and g of the map were found. If $AB = 640$ ft. and $ab = 16$ in., to what scale is the map drawn?

8. If the scale is 1 in. = 40 ft., how far is it from A to D when $ad = 20$ in.?

9. By the same scale, how far is it from C to F when $cf = 15$ in.?

10. Make a map of some small area by the use of a plane table and find the scale to which the map is drawn.

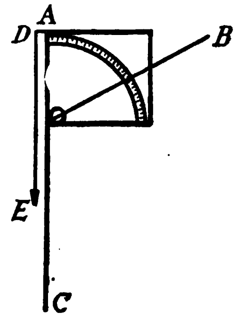


4. ANGLES OF ELEVATION AND DEPRESSION

To make a scale drawing of a height, as in finding the height of a church tower, as shown in figure, angles DBC and DAC must be known. These are called *angles of elevation*. They are the angles made by a horizontal line and the line of sight from A (or B) to the top of the tower.

1. In the figure on page 136, suppose $AB = 75$ ft., $\angle DBC = 80°$, and $\angle DAC = 50°$. Draw a figure to scale 1 in. $= 25$ ft. and by measuring CD of your drawing find the height of the spire.

2. A good substitute for a transit can be made for finding angles of elevation as follows: A large quadrant ($\frac{1}{2}$ a protractor) is fastened to a light square board which is nailed to a rod AC. DE is a plumb line to show when AC is perpendicular. OB is an arm used to mark the angle when sighting along it to the top of the object whose height is to be measured. Make such an instrument and show how to use it.



3. When the angle of elevation of the sun is 30° a building casts a shadow 60 ft. long. Make a scale drawing to a scale 1 in. $= 12$ ft., and find the height of the building.

4. By scale drawing, find the elevation of the sun when a church spire known to be 70 ft. high casts a shadow 105 ft. long.

5. A captive balloon is held by a cable 800 feet long. The cable, drawn taut owing to the wind, makes an angle with the horizontal of 65°. How high is the balloon?

6. A flagpole stands on top of a building. From a point 80 ft. from the foot of the building and on level ground, the angle of elevation to the top of the building is 35° and to the top of the flagpole 43°. Find both the height of the building and the height of the flagpole.

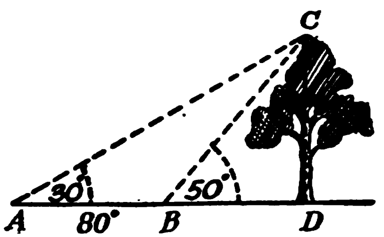
7. From a point in a high building the line of sight to an object below makes an angle of 50° with the horizontal. If

the observer is 80 ft. from the ground, how far is the object from the foot of the building?

NOTE. — This angle is called the angle of depression.

8. From the top of a mountain known to be 1750 ft. above the valley below, the angle of depression to a point in the valley below and at the foot of the mountain is 30°. How far down the slope of the mountain to the foot, assuming that the slope is gradual?

9. From the top of a hill two houses on level ground and in direct line with the observer were seen. The angle of depression of the nearer house was 40° and of the other 20°. If the houses are a mile apart (5280 ft.), the top of the hill is how many feet above the level upon which the houses stand?



10. A and B are two points 80 feet apart from which angles of elevation were taken to the top of a tree. $\angle DAC = 30°$ and $\angle DBC = 50°$. Find CD, the height of the tree.

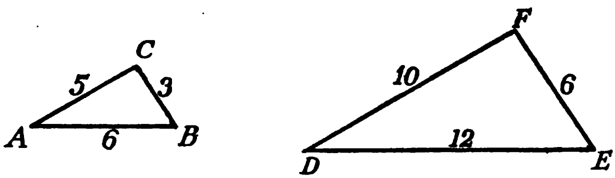
5. FINDING HEIGHTS AND DISTANCES: SIMILAR TRIANGLES

Two triangles having the same shape, whether the same size or not, are called **similar triangles**.

Thus $\triangle ABC$ and DEF are similar for they have the same shape, but they are unequal in area. The corresponding sides of $\triangle DEF$ are twice as long as those of $\triangle ABC$. But the corresponding angles you will observe to be equal. That is, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. If two angles of a triangle are known, the third angle is also

known, for the sum of all three must equal 180°. So we may say that

If two angles of one triangle are equal respectively to two angles of another triangle, the two triangles are similar

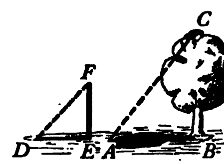


Observing the triangles above, AC is $\frac{5}{6}$ of AB, and so is DF $\frac{5}{6}$ of DE. And in general,

The ratio of any two sides of a triangle is equal to the ratio of the corresponding sides of any similar triangle.

This fact gives a method of finding heights and distances, as will be shown in the problems that follow.

1. Since the time of the ancient Greeks, the heights of objects have been found by the shadows they cast. Thales, who lived about 600 B.C., is said to have amazed the Egyptians by measuring the heights of their pyramids by this method.



Observe the triangle in the margin and tell why $\angle EDF = \angle BAC$. Why are the two triangles similar?

2. In the figure of problem 1, DE, FE, and AB can be measured. If $DE = 6$ ft. and $FE = 8$ ft., what is the ratio of FE to DE? If $AB = 60$ ft., what must CB equal?

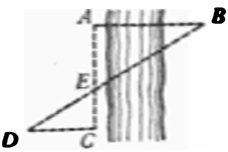
SUGGESTION.—$\frac{CB}{60} = \frac{8}{6}$.

NOTE. — By letting the unknown height or distance be represented by x, the principle of ratios of the sides of similar triangles enables one to form an equation that is solved by multiplication. Thus,

$$\frac{x}{60} = \frac{8}{6}.$$

Hence $$x = 60 \times \frac{8}{6} = 80.$$

3. When a vertical staff 4 ft. high casts a shadow 6 ft. long, how high is a tree that casts a shadow 100 ft. long?

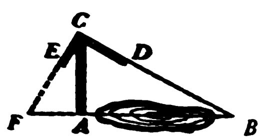


4. The distance AB across a river may be found as follows: AC is laid off along the shore at right angles to AB. Then DC is laid off at right angles to AC. Then E, the intersection of the line of sight DB with AC, is located. Study the figure and show why triangles DCE and ABE are similar.

SUGGESTION. — Remember that when two angles of one triangle are equal to two angles of the other, the triangles are similar.

5. In the figure of problem 4, show what lengths that are needed may be found. If $DC = 200$ ft., $EC = 150$ ft., and $EA = 100$ ft., how far is A from B?

6. Centuries ago, before modern instruments were invented, the following method was often used to find inaccessible distances. Upon a vertical staff AC, an instrument resembling a carpenter's square was placed. The blade CD was pointed toward B, and while held in this position, the point F on the ground toward which CE pointed was marked. In triangles AFC and ACB both have right angles. Now show why $\angle AFC = \angle ACB$.



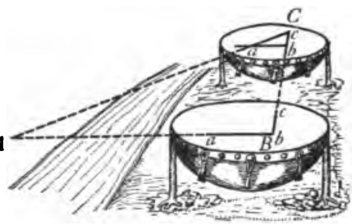
SUGGESTION. — $\angle ECD$ is a right angle. Hence $\angle s$ ACF and ACB are complements of each other, for their sum is a right angle.

7. In the figure of problem 6, what side in triangle ABC corresponds to side FA of triangle AFC? To what side does AB correspond? What lengths that may be measured are needed in order to find AB?

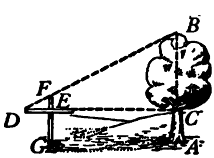
8. If $AF = 4$ ft., and $AC = 5$ ft., what is the length of AB?

9. Use the method of problem 6 to find the distance to some point, then, by making a direct measurement, find how nearly you computed the distance.

10. Before modern methods of finding distances were invented, *drum heads* were used as the *plane table* which you have already studied. The method was as follows: On a drum head placed at B a line ba was drawn toward A, and a line bc toward an accessible point C. Then the drum head was removed to C and placed with bc in the direction of BC. Then a third line ca was drawn toward A. That gave a small triangle bca similar to the large triangle BCA, for $\angle ABC = \angle abc$ and $\angle BCA$ is the same in both. Show what measures are necessary in order to find AB.



11. In problem 10, if $BC = 200$ yd., $bc = 10$ in., and $ba = 16$ in., find AB.



NOTE. — The ratios are pure abstract numbers. Thus, the ratio of bc to ba is $\frac{10}{16}$ or $\frac{5}{8}$.

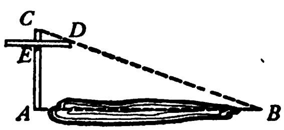
12. The *cross staff* was another instrument used for finding heights of objects. It consists of a movable crossbar DE that may be raised or lowered on the bar FG. It was used as follows: FG was erected perpendicularly at a convenient

distance from the object to be measured. Then DE was raised or lowered until D, F, and B fell in a straight line. That gave a small triangle DEF similar to triangle DCB. Show what measurements were necessary.

13. In problem 12, if $DE = 18$ in., $EF = 24$ in., $GE = 3$ ft., and $GA = 60$ ft., find the height of the tree AB.

14. Make a cross staff with arm $DE = 20$ in., and use it in finding the heights of trees, poles, or other convenient heights.

15. The figure shows how the cross staff may be used to measure horizontal distances as well as heights. Observe that triangle EDC is similar to triangle ABC, for both are right triangles and both have a common angle at C. Show what measurements are necessary in order to find AB.



16. In problem 15, find AB when $EC = 4$ in., $ED = 20$ in., and $AC = 6$ ft.

17. Use the method of problem 15 to find some short distance and test the accuracy of your computation by actual measurement.

6. FINDING HEIGHTS AND DISTANCES: TRIGONOMETRIC RATIOS

You have learned two ways of finding heights and distances: (1) By *scale drawings*, and (2) by *similar triangles*. But you found difficulties in using each method, and that slight errors in measurement caused greater errors in the results. For example, when using a scale 1 in. = 50 yd. an error of $\frac{1}{16}$ in. in measuring a line of the *plan* or *map* made an error of nearly 10 in. in the actual length represented.

Or if 1 in. = 24 mi. on a map, an error of $\frac{1}{16}$ in. would make an error of $1\frac{1}{2}$ miles in the actual distance represented.

Likewise in using the method of similar triangles we had to be *sure* that the angles of one triangle were equal to the angles of the other. Besides the factor of error that so easily entered into either of the two methods, you found that it required much time and many careful measurements to get even approximate results. Where a high degree of accuracy is desired, we must have more refined methods. Such a method has been worked out in that branch of mathematics called *trigonometry*. It is based upon the principles of similar triangles which you have studied, but these principles are applied to similar right triangles. You have already learned that any two triangles are similar when two angles of one are equal to two angles of the other. Hence

Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.

1. Draw a right triangle ABC with acute angle $A = 30°$, and right angled at B, using great care in constructing the angles. Letter the sides a, b, and c; a opposite $\angle A$, b opposite $\angle B$, and c opposite $\angle C$. Use any length you please for the sides c and b. The longer the length used, however, the more accurate the results are likely to be. Now find the ratios $\frac{a}{c}$, $\frac{c}{b}$, and $\frac{a}{b}$.

2. How did the results of the various members of the class compare? From your knowledge of similar triangles what did you expect to be true of the various results found for each ratio?

3. Let each member of the class make four right triangles with angle $A = 40°$, lettering the triangle as before, and tabulate his results in the following form :

TABLE OF RATIOS FOR $\angle A = 40°$

	LENGTH OF a	LENGTH OF b	LENGTH OF c	$a \div c$	$c \div b$	$a \div b$

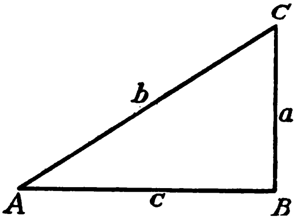
If you were very careful with your linear and angular measurements in the two problems you have solved, you found very nearly the following results :

For $\angle A = 30°$, $\frac{a}{c} = .577$, $\frac{c}{b} = .866$, $\frac{a}{b} = .5$.

For $\angle A = 40°$, $\frac{a}{c} = .839$, $\frac{c}{b} = .766$, $\frac{a}{b} = .643$.

And thus we say :

For any given acute angle of a right triangle, there is a constant ratio between any two sides, whatever their lengths.



In referring to the sides of any right triangle when dealing with either acute angle as $\angle A$, there are the "side adjacent," as c; the "side opposite," as a; and the hypotenuse, as b. The ratios of these three sides are $\frac{a}{c}$, $\frac{c}{b}$, and $\frac{a}{b}$. These three ratios are called **trigo-**

nometric ratios and are named with reference to the angle A as follows:

$$\frac{a}{c} = \frac{\text{side opposite}}{\text{side adjacent}} = \text{tangent of } \angle A \text{ (abbreviated tan } A\text{)}$$

$$\frac{c}{b} = \frac{\text{side adjacent}}{\text{hypotenuse}} = \text{cosine of } \angle A \text{ (abbreviated cos } A\text{)}$$

$$\frac{a}{b} = \frac{\text{side opposite}}{\text{hypotenuse}} = \text{sine of } \angle A \text{ (abbreviated sin } A\text{)}$$

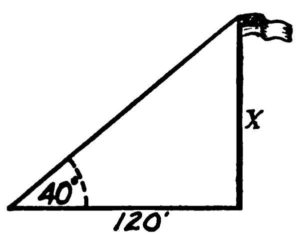
These are written as formulas

$$\tan A = \frac{a}{c}. \qquad \cos A = \frac{c}{b}. \qquad \sin A = \frac{a}{b}.$$

The values of these ratios for angles from 1° to 89° have been accurately computed and are given on the following page true to the fourth decimal place. Where great accuracy is needed, tables carried out to six, eight, or more places are used.

7. THE USE OF TRIGONOMETRIC TABLES

1. Find the height x of a flagstaff when the angle of elevation at a point 120 feet from the foot is 40°.



ANALYSIS OF THE PROBLEM. — Here the "side opposite" is unknown and the "side adjacent" known. The ratio $\frac{x}{120}$ is the *tangent ratio*. In the tables, $\tan 40° = .8391$. Hence the required equation is

$$\frac{x}{120} = .8391.$$

Solving the equation

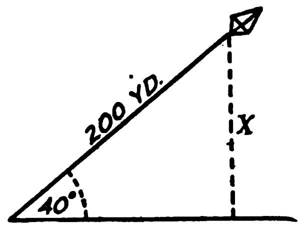
$$x = 120 \times .8391 = 100.692.$$

Hence the flagstaff is *about* 100 ft. high.

VALUES OF SINES, COSINES, AND TANGENTS

Deg.	Sine	Cosine	Tangent	Deg.	Sine	Cosine	Tangent
1	.0175	.9998	.0175	45	.7071	.7071	1.0000
2	.0349	.9994	.0349	46	.7193	.6947	1.0355
3	.0523	.9986	.0524	47	.7314	.6820	1.0724
4	.0698	.9976	.0699	48	.7431	.6691	1.1106
5	.0872	.9962	.0875	49	.7547	.6561	1.1504
6	.1045	.9945	.1051	50	.7660	.6428	1.1918
7	.1219	.9925	.1228	51	.7771	.6293	1.2349
8	.1392	.9903	.1405	52	.7880	.6157	1.2799
9	.1564	.9877	.1584	53	.7986	.6018	1.3270
10	.1736	.9848	.1763	54	.8090	.5878	1.3764
11	.1908	.9816	.1944	55	.8192	.5736	1.4281
12	.2079	.9781	.2126	56	.8290	.5592	1.4826
13	.2250	.9744	.2309	57	.8387	.5446	1.5399
14	.2419	.9703	.2493	58	.8480	.5299	1.6003
15	.2588	.9659	.2679	59	.8572	.5150	1.6643
16	.2756	.9613	.2867	60	.8660	.5000	1.7321
17	.2924	.9563	.3057	61	.8746	.4848	1.8040
18	.3090	.9511	.3249	62	.8829	.4695	1.8807
19	.3256	.9455	.3443	63	.8910	.4540	1.9626
20	.3420	.9397	.3640	64	.8988	.4384	2.0503
21	.3584	.9336	.3839	65	.9063	.4226	2.1445
22	.3746	.9272	.4040	66	.9135	.4067	2.2460
23	.3907	.9205	.4245	67	.9205	.3907	2.3559
24	.4067	.9135	.4452	68	.9272	.3746	2.4751
25	.4226	.9063	.4663	69	.9336	.3584	2.6051
26	.4384	.8988	.4877	70	.9397	.3420	2.7475
27	.4540	.8910	.5095	71	.9455	.3256	2.9042
28	.4695	.8829	.5317	72	.9511	.3090	3.0777
29	.4848	.8746	.5543	73	.9563	.2924	3.2709
30	.5000	.8660	.5774	74	.9613	.2756	3.4874
31	.5150	.8572	.6009	75	.9659	.2588	3.7321
32	.5299	.8480	.6249	76	.9703	.2419	4.0108
33	.5446	.8387	.6494	77	.9744	.2250	4.3315
34	.5592	.8290	.6745	78	.9781	.2079	4.7046
35	.5736	.8192	.7002	79	.9816	.1908	5.1446
36	.5878	.8090	.7265	80	.9848	.1736	5.6713
37	.6018	.7986	.7536	81	.9877	.1564	6.3138
38	.6157	.7880	.7813	82	.9903	.1392	7.1154
39	.6293	.7771	.8098	83	.9925	.1219	8.1443
40	.6428	.7660	.8391	84	.9945	.1045	9.5144
41	.6561	.7547	.8693	85	.9962	.0872	11.4301
42	.6691	.7431	.9004	86	.9976	.0698	14.3006
43	.6820	.7314	.9325	87	.9986	.0523	19.0811
44	.6947	.7193	.9657	88	.9994	.0349	28.6363
45	.7071	.7071	1.0000	89	.9998	.0175	57.2900

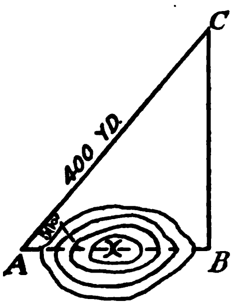
2. The length of a kite string is known to be 200 yards. When the string drawn taut by the kite makes an angle of elevation of 40°, how high is the kite?



ANALYSIS OF THE PROBLEM. — Here again the "side opposite" is unknown, but in this case the hypotenuse is known. The ratio $\frac{x}{200}$ is the *sine ratio.* In the tables, sin 40 = .6428. Hence the equation is

$$\frac{x}{200} = .6428.$$

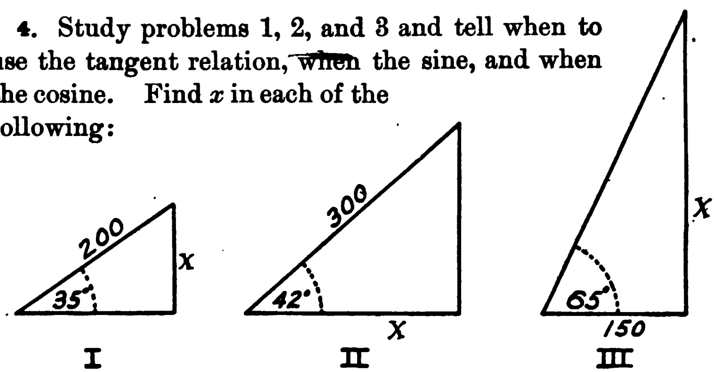
3. To find the distance directly east and west between two points A and B at opposite ends of a lake, a surveyor measures a line AC, making an angle of 50° with AB to a point C directly north of B. Find AB if $AC = 400$ yards.



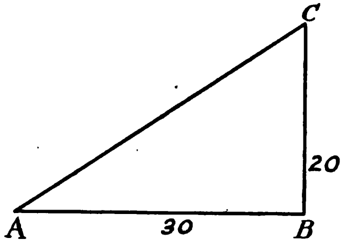
ANALYSIS OF THE PROBLEM. — Here the "side adjacent" is unknown and the hypotenuse is known. The ratio $\frac{x}{400}$ is a *cosine ratio.* In the tables, cos 50° = .6428. Hence the equation is

$$\frac{x}{400} = .6428.$$

4. Study problems 1, 2, and 3 and tell when to use the tangent relation, ~~when~~ the sine, and when the cosine. Find x in each of the following:



5. When one leg of a right angle is 20 in. and the other 30 in., what are the sizes of the two acute angles?

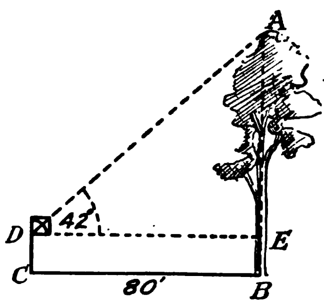


ANALYSIS OF THE PROBLEM. — In the figure, $\tan A = \frac{20}{30} = .6667$. Running down the column of tangents in the table, we find .6494 and .6745. Since .6667 lies between the two numbers that are tangents, respectively, of 33° and 34°, $\angle A$ lies between the two. Since it is nearer .6745 than .6494, we may say that $\angle A$ is *nearly* 34. In tables giving the ratios for minutes and seconds the exact size of the angle may be found.

6. When a vertical pole 12 ft. high casts a shadow $9\frac{1}{2}$ feet long, what is the angle of elevation of the sun?

7. Measure accurately the edges of a sheet of paper and by trigonometry find the angle that the diagonal makes with each edge. Now measure the angle with a protractor and see how nearly you computed the angle.

8. To find the height of a tree *AB*, a boy measured off 80 feet, and with a quadrant (described in the last chapter) found the angle of elevation to the top to be 42°, and that the quadrant at *D* was 5 ft. above the level of the ground. Find the height of the tree.

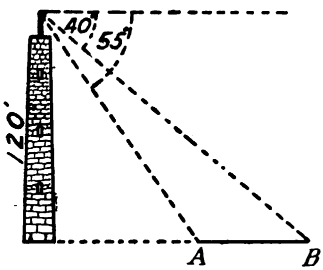


9. In the same way, measure heights in the neighborhood of the school.

10. From the 8th story of a high building known to be 112 ft. above the street, a boy observed an object at an angle

of depression of 30°. How far was the object from the building?

11. From the top of a tower known to be 120 ft. high, a boy took the angle of depression of two trees which lay in direct line upon the same horizontal plane with the tower. The angle of the nearer was 55° and that of the other 40°. How far apart were they?



12. The distance from the base to the top of a hill up a uniform incline of 36° is 300 yards. How far is the top above the line of the base?

13. A kite string 1000 feet long makes an angle of 65° with the horizontal. Not allowing for sag, how high is the kite? How far is it from the holder of the string to a point directly under the kite?

14. From the top of a cliff 200 feet above the level of the water, a boat is seen at an angle of depression of 20°. How far is the boat from the foot of the cliff?

15. When an airplane was directly over point A, an observer at point B 500 yards away observed the angle of elevation to be 53°. How high was the plane?

16. From the top of a cliff 175 feet high on the bank of a river, the angle of depression of the opposite side was 28°. How wide was the river?

17. When the angle of elevation of the sun is 36°, how long a shadow on level ground will a pole 80 feet tall cast?

$$\frac{80}{x} = \tan 36°.$$

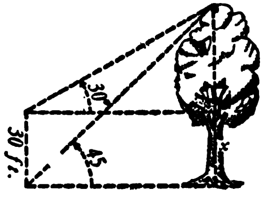
CHAPTER IX

EQUATIONS WITH TWO UNKNOWNS

IN the problems that you have met in the preceding chapters you have had equations involving but one unknown number; or, at least, you have stated all of the unknowns in terms of a single unknown. In many of the problems of mathematics, there are two or more unknown numbers. In fact, many problems arising in finding heights and distances by use of trigonometric ratios, which you have just studied, involve two unknown numbers. We shall see in this chapter how to solve such problems.

1. FINDING HEIGHTS AND DISTANCES

1. The angle of elevation of a tree is 45° from a point on the ground, and from an upper window 30 feet above the point on the ground it is only 30°. Find the height of the tree.



ANALYSIS OF THE PROBLEM. — Studying the figure, you observe that in each of the right triangles the two sides needed to express a ratio are unknown. Calling x the height of the tree, and the "side adjacent" to the known angle y, we can form the following equations:

$$\frac{x}{y} = \tan 45°.$$

$$\frac{x-30}{y} = \tan 30°.$$

These two equations are said to form a pair of **simultaneous equations in two unknowns.** To solve them, we must in some way combine them into one equation that will have but one unknown. This can be done in several ways. In this particular problem, perhaps the simplest way is to "solve for y in terms of x" in the first equation and "substitute the value found" in the second equation.

Since $\tan 45° = 1$ and $\tan 30° = .5774$, the equations become

1. $$\frac{x}{y} = 1.$$

2. $$\frac{x - 30}{y} = .5774.$$

Multiplying equation 1 by y,

3. $$x = y.$$

Substituting x for y in equation 2,

4. $$\frac{x - 30}{x} = .5774.$$

We now have one equation having but one unknown and can solve it by the methods already studied.

Multiplying both sides by x,

5. $$x - 30 = .5774\,x.$$

Adding 30 to both sides,

6. $$x = 30 + .5774\,x.$$

Subtracting $.5774\,x$ from both sides,

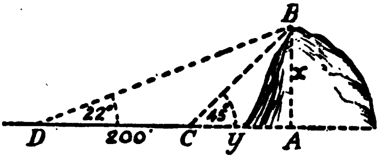
7. $$.4226\,x = 30.$$

Dividing both sides by $.4226$,

8. $$x = 71, \textit{ nearly}.$$

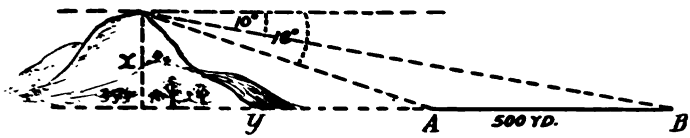
This method of solving a pair of simultaneous equations is called the **method by substitution.**

2. To find the height of a hill above the level, the following measurements were taken: From C the angle of elevation of the top of the hill was found to be 45°; from D, a point 200 feet farther from the foot of the hill, the



angle of elevation of the top was only 22°. Find the height of the hill.

3. From the top of a hill two houses in line with the observer and known to be 500 yards apart were seen. The



angle of depression of the nearer house was 18°, and of the other 10°. Find the height of the hill.

1. $$\frac{y}{x} = \tan 72°.$$ Why?

2. $$\frac{y + 500}{x} = \tan 80°.$$ Why?

Observe that equation 2 may be written

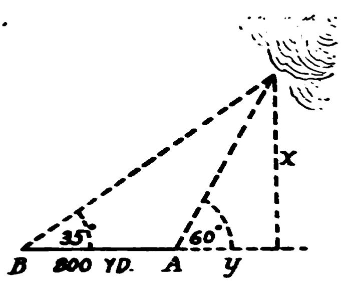
3. $$\frac{y}{x} + \frac{500}{x} = \tan 80°.$$ Why?

Subtracting equation 1 from equation 3,

4. $$\frac{500}{x} = \tan 80° - \tan 72°.$$

We have thus eliminated y and have one equation in one unknown.

This method of elimination is called **elimination by subtraction.**

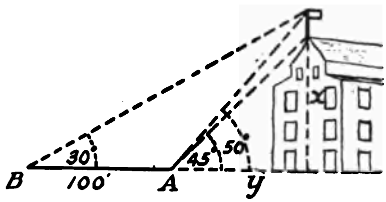


In equation 4, substitute the value of tan 80° and tan 72° and solve for x.

4. To find the height of a cloud. I observed its elevation to be 35°. I then walked toward it, on level ground, 800

yards and found the elevation to be 60°. Find the height of the cloud.

5. To find the length of a flagpole on the top of a tall building, I took the following measurements: From a point A, from which I was not able to measure to the foot of the building, I found the angle of elevation of the top of the flagpole to be 50°; and I found the angle of elevation to the top of the building to be 45°, so that the distance to the bottom was equal to the height. From a point 100 feet farther away the angle of elevation to the top of the flagpole was 30°. Find the height of the flagpole and the height of the building.



2. GENERAL METHODS OF ELIMINATION

In the five problems that you have just solved, you have had to combine the two equations in such a way as to get a single equation having but one unknown. This process was called **elimination.** We shall now discuss the general methods of elimination and give drill in using them.

3. ELIMINATION BY ADDITION AND SUBTRACTION

The two problems that follow illustrate **elimination by addition and subtraction.**

1. Solve

$$2x + y = 11, \tag{1}$$
$$5x - 2y = 14. \tag{2}$$

Multiplying equation (1) by 2 so as to make the coefficient of y alike numerically, we have

$$4x + 2y = 22 \quad (3)$$
$$5x - 2y = 14 \quad (4)$$

Adding (3) and (4), $9x = 36$ (5)

Dividing by 9, $x = 4.$

Substituting 4 for x in equation (1).

$$8 + y = 11.$$

Solving for y, $y = 3.$

Check by substituting these values in equations (1) and (2),

$$8 + 3 = 11,$$
$$20 - 6 = 14.$$

This method is called **elimination by addition.**

2. Solve $2x + 3y = 12,$ (1)

$4x - y = 10.$ (2)

Multiplying equation (1) by 2 to make the coefficients of x alike, we have

$$4x + 6y = 24 \quad (3)$$
$$4x - y = 10 \quad (4)$$

Subtracting (4) from (3), $7y = 14$ (5)

Dividing by 7, $y = 2.$

Substituting 2 for y in equation (1),

$$2x + 6 = 12.$$

Solving for x, $x = 3.$

Check by substituting these values in equation (1) and (2),

$$6 + 6 = 12,$$
$$12 - 2 = 10.$$

This method is called **elimination by subtraction.**

Outline of the Method

1. *Multiply the members of both equations, if necessary, by such numbers as will make the coefficients of one of the unknowns the same numerically in both equations.*

2. *If the like coefficients have different signs, add the two equations; if they have the same sign, subtract one equation from the other. This gives one equation having but one unknown.*

3. *Solve the resulting equation for the unknown which it contains.*

4. *Substitute the value of the unknown found in step 3 in either equation containing both unknowns. This gives an equation containing the second unknown. Solve the equation for the second unknown.*

5. *Check by substituting the values found, in both of the given equations.*

Eliminate by addition or subtraction, solve and check:

1. $\begin{cases} x + y = 7, \\ x - y = -3. \end{cases}$

2. $\begin{cases} 5x + 4y = 22, \\ 3x + y = 9. \end{cases}$

3. $\begin{cases} 6x - y = 10, \\ 7x - 2y = 15. \end{cases}$

4. $\begin{cases} 7x - 2y = -15, \\ -2x + y = 6. \end{cases}$

5. $\begin{cases} 4x - 3y = 1, \\ 3x - 4y = 6. \end{cases}$

6. $\begin{cases} 2x + 5y = 15, \\ 3x - 4y = 11. \end{cases}$

7. $\begin{cases} x + 4y = 12, \\ 2x - y = 6. \end{cases}$

8. $\begin{cases} 6x + 7y = -33, \\ 5x - 4y = 2. \end{cases}$

9. $\begin{cases} 7x + y = 42, \\ 3x - y = 8. \end{cases}$

10. $\begin{cases} 5x - 2y = 35, \\ 6x + y = 25. \end{cases}$

11. $\begin{cases} 2x - y = 5, \\ 5x - 2y = 14. \end{cases}$

12. $\begin{cases} 5a - 2b = 1, \\ 8a - 5b = -11. \end{cases}$

13. $\begin{cases} 2x + y = 35, \\ 5x - 3y = 27. \end{cases}$

14. $\begin{cases} 3x + 2y = 4, \\ 3x - 4y = 1. \end{cases}$

15. $\begin{cases} 3x - 4y = 45, \\ 2x + 6y = 4. \end{cases}$

16. $\begin{cases} 3x - y = 5, \\ 5x + 2y = 23. \end{cases}$

17. $\begin{cases} 3x + 2y = 4, \\ 2x + 7y = 31. \end{cases}$

18. $\begin{cases} x - 3y = 3, \\ 2x + y = 13. \end{cases}$

19. $\begin{cases} 3x - 2y = 9, \\ 2x + 3y = 19. \end{cases}$

20. $\begin{cases} 3x + y = 16, \\ 2x - 3y = 7. \end{cases}$

4. PROBLEMS INVOLVING TWO UNKNOWNS

You saw a real need of knowing how to state and solve problems involving two unknowns in the problems at the beginning of this chapter. The following problems are given more for practice in stating and solving problems with two unknowns than to meet a real need. When there are two unknowns, there must be two conditions, each of which gives an equation in the two unknowns.

1. A 12-foot post is to be set in the ground so that the part above ground is 3 feet longer than the other part. Find the depth to which the hole must be dug.

Let $x =$ the length in the ground, and $y =$ the part above ground.

By the first condition, $x + y = 12.$ (1)

By the second condition, $y - x = 3.$ (2)

2. A rectangular field is 10 rd. longer than it is wide, and its perimeter is 100 rd. Find its dimensions.

Let $x =$ length, and $y =$ width.

By the first condition, $x - y = 10.$ (1)

By the second condition, $2x + 2y = 100.$ (2)

3. A rectangle whose length is twice its width has a perimeter of 120 feet. Find its dimensions.

4. A board 10 feet long is to be cut into two pieces one of which is one foot longer than the other. Find the length of each part.

5. A board 14 feet long is to be cut into two parts so that one piece is $\frac{3}{4}$ of the other. Find the length of each part.

6. In a classroom of 42 pupils there are 8 more girls than boys. How many of each?

7. In a high school having 780 pupils in both the Junior and Senior schools, there are only $\frac{2}{3}$ as many in the Senior high school as in the Junior high school. How many in each?

8. A grocer blended tea costing 24¢ per pound with tea costing 30¢, and sold it all for 40¢ per pound. If the total cost was \$5.28 and the profit \$2.72, how much of each kind did he use?

9. A boy has \$3.50 in nickels and dimes. There are 50 coins in all. How many of each has he?

10. One acute angle of a right triangle is 16° larger than the other. Find each acute angle.

5. ELIMINATION BY SUBSTITUTION

The following problem illustrates the method of **elimination by substitution.**

1. Solve $x + y = 5,$ (1)

$x + 3y = 9.$ (2)

To eliminate y, solve (1) for y in terms of x.

Solving (1) for y, $y = 5 - x.$ (3)

Substituting this value of y in equation (2),

$$x + 3(5 - x) = 9, \qquad (4)$$
$$x + 15 - 3x = 9,$$
$$-2x = -6,$$
$$x = 3.$$

Substituting 3 for x in equation (1),

$$3 + y = 5,$$
$$y = 2.$$

Check by substituting these values in equations (1) and (2),

$$3 + 2 = 5,$$
$$3 + 6 = 9.$$

Outline of the Method

1. *Solve one equation for one unknown in terms of the other.*

2. *Substitute the value found in place of the unknown in the other equation. This gives an equation having but one unknown.*

3. *Solve the resulting equation for the unknown which it contains.*

4. *Substitute the value of the unknown then found in either of the equations containing two unknowns and solve for the second unknown.*

5. *Check by substituting the values found in the given equations.*

Eliminate by substitution, solve, and check:

1. $\begin{cases} x + y = 5, \\ x + 5y = 13. \end{cases}$

2. $\begin{cases} x - y = 4, \\ x + 2y = 16. \end{cases}$

3. $\begin{cases} x + 4y = 7, \\ x + 6y = 9. \end{cases}$

4. $\begin{cases} x + y = 30, \\ 3x - 2y = 25. \end{cases}$

5. $\begin{cases} x - y = 4, \\ 2x + y = 14. \end{cases}$

6. $\begin{cases} 2x + y = 7, \\ x + 3y = 16. \end{cases}$

7. $\begin{cases} 3x - 7y = 40, \\ 4x - 3y = 9. \end{cases}$

8. $\begin{cases} 4x - 5y = 26, \\ 3x - 6y = 15. \end{cases}$

9. $\begin{cases} 3a + 7b = 16, \\ 2a + 5b = 13. \end{cases}$

10. $\begin{cases} 3a + 2b = 26, \\ 5a - 2b = 38. \end{cases}$

11. $\begin{cases} r + 2s = 4, \\ 3r - s = 5. \end{cases}$

12. $\begin{cases} r + 2s = 4, \\ r - 2s = 12. \end{cases}$

6. PROBLEMS OF TWO UNKNOWNS

1. Find two numbers whose sum is 21 and whose difference is 9.

2. The sum of two numbers is 13. One exceeds twice the other by 1. What are the numbers?

3. The difference of two numbers is 72. One of them is 4 times the other. What are the numbers?

4. The sum of two numbers is 64. The quotient of the larger divided by the smaller is 7. What are the numbers?

5. The value of a fraction is $\frac{2}{3}$. If 7 is added to each term, the value of the resulting fraction is $\frac{3}{4}$. Find the fraction.

6. A surveyor wishes to set a stake in a line 200 feet long so that it will divide the line in parts whose ratio is as 3 to 4. How far from each end must it be set?

7. A boy has 50 coins in dimes and nickels, making a total of $3. How many of each has he?

8. One of two supplementary angles is 20° more than twice the other. How many degrees in each?

9. If one acute angle of a right triangle is twice the other, how many degrees in each?

10. The base angles of an isosceles triangle are each twice the vertical angle. How many degrees in each angle of the triangle?

11. How much milk testing 4% butter fat and cream testing 24% butter fat must be mixed to make 20 gallons that test 20% butter fat?

12. A number of two digits whose sum is 11 is diminished by 45 when the digits are interchanged. What is the number?

SUGGESTION. — Let t = tens' digit and u = ones' digit. Then the number is $10t + u$.

13. The sum of two digits of a number is 9. By interchanging the digits the number formed is 45 larger than the given number. Find it.

7. SYSTEMS OF EQUATIONS CONTAINING FRACTIONS

There is a type of problem giving equations whose unknowns occur only as monomial denominators. These are solved by eliminating one unknown before clearing the equation of fractions. The type of problem and the method of solving it is here shown.

1. A tank can be filled by two pipes one running two hours and the other four hours, or by the first running three hours and the other two hours. How long will it take each alone?

SOLUTION

Let x = time required by first alone.

And y = time required by second alone.

Then $\frac{1}{x}$ = the part of the cistern filled by the first in one hour.

And $\frac{1}{y}$ = the part of the cistern filled by the second in one hour.

From the first condition, $\frac{2}{x}+\frac{4}{y}=1.$ (1)

From the second condition, $\frac{3}{x}+\frac{2}{y}=1.$ (2)

Multiplying equation (2) by 2 and subtracting equation (1),

$$\frac{6}{x}+\frac{4}{y}=2 \qquad (3)$$

$$\frac{2}{x}+\frac{4}{y}=1 \qquad (1)$$

$$\frac{4}{x}\qquad=1 \qquad (4)$$

Solving (4), $x=4.$

Substituting in (1), $\frac{2}{4}+\frac{4}{y}=1.$ (5)

Solving (5), $y=8.$

Solve and check:

1. $\begin{cases}\frac{1}{x}+\frac{2}{y}=2,\\ \frac{2}{x}-\frac{2}{y}=1.\end{cases}$

2. $\begin{cases}\frac{4}{x}+\frac{9}{y}=1,\\ \frac{3}{x}+\frac{6}{y}=\frac{1}{2}.\end{cases}$

3. $\begin{cases}\frac{1}{x}+\frac{1}{y}=7,\\ \frac{1}{x}-\frac{1}{y}=3.\end{cases}$

4. $\begin{cases}\frac{3}{x}+\frac{8}{y}=3,\\ \frac{15}{x}-\frac{4}{y}=4.\end{cases}$

5. $\begin{cases}\frac{3}{a}+\frac{4}{b}=3,\\ \frac{6}{a}-\frac{2}{b}=1.\end{cases}$

6. $\begin{cases}\frac{6}{a}-\frac{10}{b}=1,\\ \frac{2}{a}+\frac{15}{b}=4.\end{cases}$

7. $\begin{cases}\frac{6}{r}+\frac{8}{s}=3,\\ \frac{12}{r}-\frac{20}{s}=-3.\end{cases}$

8. $\begin{cases}\frac{1}{2x}-\frac{2}{5y}=4,\\ \frac{5}{x}+\frac{6}{y}=7.\end{cases}$

8. WORK PROBLEMS OF TWO UNKNOWNS

1. A tank can be $\frac{7}{12}$ filled by one pipe running one hour and another running two hours. But it can be entirely filled by the first running two hours and the second three hours. How long will it take each alone?

2. A piece of work can be completed by two men each working $1\frac{7}{8}$ days; or by one working three days and the other $1\frac{1}{5}$ days. How long will it take each alone?

3. A tank can be filled by two pipes, one running six hours and the other three hours; or by the first running four hours and the second running six hours. How long will it take each alone?

4. A water tank can be filled by two pipes in $8\frac{8}{9}$ minutes. If the first is left open 10 minutes and the second 8 minutes, the tank will be filled. In what time can each pipe alone fill the tank?

5. Two steam pumps together can fill a reservoir with water in $6\frac{3}{4}$ hours. If one pump works 8 hours and the other 3 hours 48 minutes, the reservoir will be filled. How long would it take each pump alone to fill it?

6. In a factory which operates two sizes of machines it is found that two large machines and five small ones can turn out a certain quantity of goods in twelve hours, while four large ones and three small ones can do it in ten hours. Find how long it would require one machine of each size alone to turn out the goods.

7. A and B together can do a piece of work in $13\frac{1}{3}$ days. After they have worked six days B leaves, and A finishes the work in $16\frac{1}{2}$ days more. In how many days could each of them alone do the work?

CHAPTER X

GRAPHIC REPRESENTATION OF EQUATIONS IN TWO UNKNOWNS

EQUATIONS containing two unknowns may be pictured to the eye by means of lines. These lines show relationships more clearly than the equation itself shows them. Since a line may be considered as a series of points, the first step in learning to represent an equation by **graphs** is to learn how to locate points.

1. HOW TO LOCATE POINTS

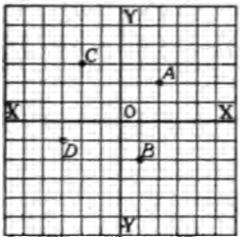
As you know, points on the earth's surface are definitely located by their latitude and longtitude. Thus, Chicago is 88° west of the prime meridian and 42° north of the equator. Likewise, any point in a plane may be located by knowing its distance from each of two intersecting straight lines.

For convenience in locating points we take two straight intersecting lines, one vertical and one horizontal, which make right angles with each other. These are called the **axes** and their intersection is called the **origin.**

The vertical axis is usually called the **y-axis,** and the horizontal axis, the **x-axis.**

Point A may be described as 2 units to the *right* of the y-axis, and 2 units *above* the x-axis.

To avoid saying "to the left of," "to the right of," "above" or "below," it has been agreed to call distances to the *right* of the y-axis **positive,** and to the left of it **negative;**



and to call distances *above* the x-axis *positive*, and *below* it *negative*.

Thus, the position of B is $(+1, \ -2)$ which means that it is 1 unit to the right of the y-axis, and 2 units below the x-axis.

To avoid confusion, the distance to the right or left of the y-axis is always given *first*, and the distance below or above the x-axis is given *second*.

The first is called the **x-distance** and the other the **y-distance**.

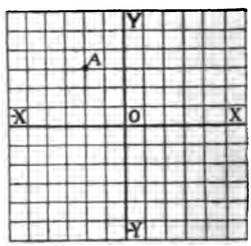
1. Give the position of A in terms of signed numbers.

2. In the same way give the location of C. Of D.

3. Locate a point described as $(+3, \ -4)$.

2. PLOTTING A POINT

"Plotting a point" means locating it on squared paper when its distances from the axes are known. Thus, to plot point A whose distances are described as $(-2, \ +3)$ is to count 2 units to the *left* on the x-axis, and 3 units *above* on a line parallel to the y-axis.



1. On squared paper locate a point 4 units to the left and 3 units above the y-axis and the x-axis, respectively. Give the method of describing this with signed numbers.

2. The x-distance of a point is -3, and its y-distance is $+2$. Locate it.

3. A certain point is on the y-axis. Can you definitely locate it?

4. If the y-distance is zero, upon what axis is the point?

5. What represents the x-distance of a point on the y-axis?

6. Locate the point whose position is $(0, 3)$; $(-2, 0)$.

7. Plot the following points: $(2, 3)$, $(3, -2)$, $(-1, -3)$, $(0, -3)$, $(2, 0)$.

8. Plot the following points: $(1\frac{1}{2}, \frac{1}{4})$, $(-\frac{1}{2}, +1\frac{1}{2})$, $(-1\frac{1}{4}, +\frac{1}{4})$.

3. PICTURING AN EQUATION OF TWO UNKNOWNS

Now that we can locate points, we can picture an equation. Thus we shall picture the equation $3x + y = 6$.

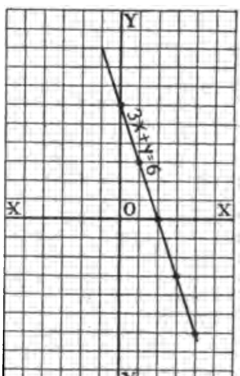
You have seen that if you solve for either unknown, the result contains the other unknown. Thus, to solve for y,

$$y = 6 - 3x.$$

Thus it is seen that the value of y depends upon the value of x. That is, y is a **function** of x, or x is a *function* of y.

In the above solution for y in terms of x, if $x = 1$, $y = 3$; if $x = 2$, $y = 0$; if $x = 3$, $y = -3$; if $x = 4$, $y = -6$; etc., giving a new value for y for any value given x.

By carefully plotting the points represented by these solutions, calling the x-value the x-distance, and the y-value the y-distance, we get a series of points lying on a straight line.

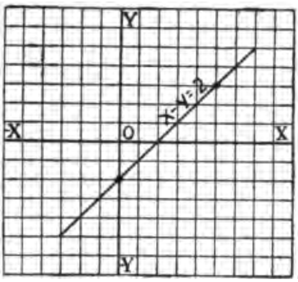


This straight line is the **graphic representation** of the equation $3x + y = 6$.

Since the graphs of all equations of two unknowns, each of which is of the first power or degree, can be shown to be straight lines, such equations are called **linear equations.**

Since two points determine a straight line, by plotting two points, the graph of a linear equation can be drawn.

1. Draw a graph of the equation $x - y = 2$.



Solving for y,

$$y = x - 2$$

When $x = 0,\ y = -2,$

When $x = 5,\ y = 3.$

NOTE. — By locating points which are several units apart, an accurate line is more easily drawn. That is why $x = 5$ instead of $x = 1$ was taken.

Plotting the points $(0, -2)$ and $(5, 3)$ a straight line is drawn through these points by means of a ruler.

2. In the equation $3x + 2y = 18$, what is the value of y when $x = 0$? What is the value of x when $y = 0$?

From the two sets of values, $\begin{cases} x = 0 \\ y = 9 \end{cases}$ and $\begin{cases} x = 6 \\ y = 0 \end{cases}$, plot the points $(0, 9)$ and $(6, 0)$ and draw the graph of the equation.

Observe that the two sets of values needed are more easily found as in problem 2 than as in problem 1.

That is, letting $x = 0$ and $y = 0$ we find the points where the line (graph) cuts the y- and x-axes.

3. Plot the equation $x + 2y = 8$ by finding where the line cuts the axes.

4. Where does the graph of the equation $4x + 3y = 12$ cut the x-axis? The y-axis?

5. Plot the equation $2x - 3y = 6$.

6. Plot the equation $3x - y = 9$.

4. THE SOLUTION OF A LINEAR EQUATION OF TWO UNKNOWNS

In the equation $2x + 3 = 7$ you found *but one* value of x that satisfied the equation. That was $x = 2$. This is called **a solution** of the equation.

But in $2x + 3y = 6$ you found *many* solutions. In fact, for *any* value given one unknown there was a value for the other. Thus, any set of values that satisfied the equation was a *solution* of the equation. This shows that there is no definite solution for an equation in two unknowns having but one equation.

1. Twice a given number is 6 more than 3 times another. What are the numbers?

Analysis of the Problem.—There is only one condition given, hence there can be only one equation. The equation is $2x - 3y = 6$. Draw a graph of the equation and show a number of sets of values that satisfy the equation.

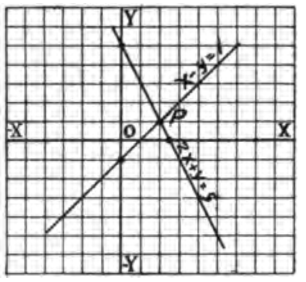
5. THE SOLUTION OF A SYSTEM OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

In the problems and equations that you had in the last chapter, you were given two equations or a problem with two conditions from which you could write two equations. The system you solved had but one set of solutions. This is shown clearly by drawing a graph of each equation. The **solution of the system** is the point where the two lines intersect.

1. Draw a graph of the system $\begin{cases} x - y = 1, \\ 2x + y = 5. \end{cases}$

By measuring, observe that point P is described as (2, 1), and if you should solve the system by the methods of elimination, you would find $x = 2$, $y = 1$.

(2, 1) are called the **coördinates** of point P. The solution of a system of linear equations, then, is the coördinates of the point where the lines intersect.



Solve the following graphically, and check by elimination:

2. $\begin{cases} x+y=5, \\ 2x-y=1. \end{cases}$

3. $\begin{cases} x+\ \ y=1, \\ 3x-4y=10. \end{cases}$

4. $\begin{cases} x-3y=1, \\ x+\ \ y=-3. \end{cases}$

5. $\begin{cases} x-y=2, \\ 2x+y=7. \end{cases}$

6. $\begin{cases} x-\ \ y=2, \\ x+2y=8. \end{cases}$

7. $\begin{cases} x-\ \ y=5, \\ x+2y=-1. \end{cases}$

8. $\begin{cases} 2x+\ \ y=-1, \\ x-3y=10. \end{cases}$

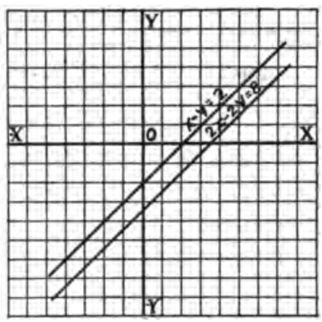
9. $\begin{cases} x+y=1, \\ 2x-y=8. \end{cases}$

10. Show by their graphs that $2x+y=11$ and $5x-2y=5$ intersect at a point whose coördinates are (2, 5), and hence form a system whose solution is $x=2$, $y=5$.

Two equations in two unknowns that have but one solution (that is, two equations whose graphs intersect) are called a system of **consistent linear equations.** They are called also a system of **simultaneous equations,** for a set of solutions satisfies both equations at the same time.

6. INCONSISTENT EQUATIONS: PARALLEL LINES

The figure in the margin shows the graphs of the two equations,
$$\begin{cases} x - y = 2, \\ 2x - 2y = 7. \end{cases}$$



Observe that these two lines do not intersect but are parallel. It is evident that they have no point in common. That is, they do not intersect, and hence they have no common solution. If you undertake to solve by elimination, both unknowns are eliminated at once and you get no solution. Such a system is called an **inconsistent system**, to distinguish it from the simultaneous equations just studied.

Plot graphs of the following:

1. $\begin{cases} x - y = 2, \\ 2x - 2y = 8. \end{cases}$

2. $\begin{cases} x + 2y = 5, \\ 2x + 4y = 1. \end{cases}$

3. $\begin{cases} 2x - y = 4, \\ 6x - 3y = 2. \end{cases}$

4. $\begin{cases} 2x + y = 1, \\ 8x + 4y = 10. \end{cases}$

5. $\begin{cases} x - 3y = 5, \\ 2x - 6y = 1. \end{cases}$

6. $\begin{cases} x = 5 - y, \\ 2y = 10 - 2x. \end{cases}$

7. Show by their graphs that $5x - y = 2$ and $5x - y = 10$ form an inconsistent system.

8. Show by their graphs that $3x - y = 6$ and $x + 2y = 4$ form a consistent or simultaneous system of equations.

9. Find by their graphs whether $5x - y = 0$ and $3y = 40 + 15x$ are consistent or inconsistent.

CHAPTER XI

ROOTS AND RADICALS

1. THE MEANING OF ROOTS

You have learned the meaning of squaring a number or of raising it to any power. Problems arise in mathematics in which it is necessary to find the number which squared gave a certain number.

The number which squared gave a certain number is called the **square root** of that number.

1. What number squared, or multiplied by itself, gives 9? 16? 25? 49? a^2? x^2?

2. Since either $(+3)^2$ or $(-3)^2$ gives $+9$, how many answers are there to the first question?

3. What is the *square root* of 64? Of 81? Of y^2? Of $16x^2$? Of $25y^4$?

4. The positive square root of a number is indicated by the sign ($\sqrt{\quad}$) called the **radical sign**. The negative square root is denoted by ($-\sqrt{\quad}$). What is $\sqrt{9}$? What is $-\sqrt{9}$?

5. Give the value of: $\sqrt{100}$; $-\sqrt{49}$; $\sqrt{144}$; $-\sqrt{25x^2}$.

You have seen that

Every positive number has two square roots, which have the same absolute value, one positive and one negative.

The two square roots of a positive number are often written with a double sign. Thus, $\sqrt{9} = \pm 3$, $\sqrt{16} = \pm 4$, $\sqrt{25} = \pm 5$, etc.

To solve $y^2 = 25$ is to find what number squared will give 25. As you have seen, there are two answers. Either -5 or $+5$ when squared gives 25. The solution can be written $y = -\sqrt{25}$ or $\sqrt{25} = -5$ or 5. But this is more commonly written $y = \pm\sqrt{25} = \pm 5$.

Since both negative and positive numbers when squared give a positive number, it is seen that a negative number has no square root that can be expressed either as a positive or negative number.

NOTE. — The square root of a negative number is called an **imaginary number**, and is not discussed in this book.

Give at sight the square roots of:

6. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 144.
7. a^2, b^4, c^2, d^6, e^8, x^{10}, y^{12}, z^{20}.
8. $49\,y^6$, $25\,a^4$, $36\,m^8$, $64\,z^{10}$, $81\,n^{20}$.
9. a^4b^6, a^6c^{10}, x^8y^{12}, m^4n^{10}, $z^{10}y^{20}$.
10. $4\,x^2y^2$, $16\,m^2n^4$, $36\,m^4n^6$, $49\,x^6y^8$.

2. EQUATIONS SOLVED BY FINDING SQUARE ROOTS

Many equations arise in which the solution is found by finding the square root of a number. Thus, to solve $x^2 = 9$, $x = \pm\sqrt{9} = \pm 3$.

Solve and check:

1. $x^2 = 25$.
2. $x^2 = 81$.
3. $x^2 = 36$.
4. $x^2 = 16$.
5. $x^2 - 64 = 0$.
6. $x^2 - 121 = 0$.
7. $x^2 - 225 = 0$.
8. $x^2 - 100 = 0$.
9. $2\,x^2 = 8$.
10. $7\,x^2 = 28$.
11. $3\,x^2 = 75$.
12. $5\,x^2 = 405$.
13. The area of a square garden is 144 sq. ft. What is the side of the square?

14. The area of a circle when using $\frac{22}{7}$ as the value of π was 154 sq. ft. Find the radius.

15. The area of a rectangular garden twice as long as wide is 450 sq. ft. What are its dimensions?

3. QUADRATIC SURDS

The indicated square root of a positive number is called a **quadratic surd.** Thus, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{x}$, and $\sqrt{2a}$ are quadratic surds.

You learned in **Book II** how to find the *approximate* root of such numbers to any required degree of accuracy. Thus, you found $\sqrt{5} = 2.236$ to the nearest thousandth.

4. SIMPLIFYING QUADRATIC SURDS

Large numbers under the radical sign may be simplified by removing factors that are perfect squares. This may leave a surd that is so simple that its value is known or may be found by tables. Thus,

$$\sqrt{405} = \sqrt{81 \times 5} = 9\sqrt{5} = 9 \times 2.236.$$

The principle used above is that

The square root of a product is the same as the product of the square roots of each of its factors.

Or as a formula,

$$\sqrt{abc} = \sqrt{a} \times \sqrt{b} \times \sqrt{c}$$

Thus,

$$\sqrt{2880} = \sqrt{6 \cdot 6 \cdot 4 \cdot 4 \cdot 5} = \sqrt{36 \cdot 16 \cdot 5} = 6 \cdot 4 \cdot \sqrt{5} = 24 \times 2.236.$$

Simplify the exercises on the following page so as to leave no factor that is a perfect square under the radical sign.

1.	$\sqrt{75}$.	5.	$\sqrt{125}$.	9.	$\sqrt{a^3}$.	13.	$\sqrt{8a^8}$.
2.	$\sqrt{27}$.	6.	$\sqrt{180}$.	10.	$\sqrt{x^5}$.	14.	$\sqrt{32x^5}$.
3.	$\sqrt{50}$.	7.	$\sqrt{175}$.	11.	$\sqrt{b^7}$.	15.	$\sqrt{28y^3}$.
4.	$\sqrt{40}$.	8.	$\sqrt{300}$.	12.	$\sqrt{c^9}$.	16.	$\sqrt{12a^3b^5}$.

5. THE VALUE OF KNOWING THE APPROXIMATE ROOT OF CERTAIN SURDS

You saw in the above exercises that by simplifying surds they reduce to surds of smaller numbers. When the value of these smaller surds is known, either through memory or from tables, the work of finding the square root of a number as in **Book II** can be saved.

Using the following table, find the value of the surds given below:

$\sqrt{2} = 1.414$	$\sqrt{6} = 2.449$
$\sqrt{3} = 1.732$	$\sqrt{7} = 2.646$
$\sqrt{5} = 2.236$	$\sqrt{10} = 3.162$

NOTE. — This table and the following exercises are to show the value of simplifying surds. A more complete table is given on page 179.

1.	$\sqrt{75}$.	6.	$\sqrt{216}$.	11.	$\sqrt{250}$.	16.	$\sqrt{7500}$.
2.	$\sqrt{80}$.	7.	$\sqrt{175}$.	12.	$\sqrt{448}$.	17.	$\sqrt{3200}$.
3.	$\sqrt{90}$.	8.	$\sqrt{162}$.	13.	$\sqrt{243}$.	18.	$\sqrt{4800}$.
4.	$\sqrt{300}$.	9.	$\sqrt{245}$.	14.	$\sqrt{256}$.	19.	$\sqrt{2800}$.
5.	$\sqrt{720}$.	10.	$\sqrt{360}$.	15.	$\sqrt{1200}$.	20.	$\sqrt{9600}$.

21. Find the value of $\sqrt{50} + \sqrt{128}$.

$$\sqrt{50} = 5\sqrt{2}; \ \sqrt{128} = 8\sqrt{2}.$$

Hence $\sqrt{50} + \sqrt{128} = 5\sqrt{2} + 8\sqrt{2} = 13\sqrt{2} = 13 \times 1.414$.

22. Find the value of $\sqrt{48} + \sqrt{243} - \sqrt{75}$.

23. Find the value of $\sqrt{125} + \sqrt{245} - \sqrt{80}$.

6. RATIONALIZING THE DENOMINATOR OF A FRACTION

As you know, a fraction is squared by squaring its numerator and denominator separately. Thus,

$$\left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}. \quad \text{Hence } \sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}.$$

$$\sqrt{\frac{2}{3}} \doteq \frac{\sqrt{2}}{\sqrt{3}} = \frac{1.414}{1.732} = 1.414 \div 1.732.$$

The division by 1.732 can be saved by making the denominator a perfect square before taking the square root. Thus, $\frac{2}{3} = \frac{6}{9}$. (Why?)

Hence $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{\sqrt{9}} = \frac{2.449}{3} = .816$, which can be done mentally.

1. Using the tables of the last section, find the value of $\sqrt{\frac{2}{5}}$.

SOLUTION

$$\sqrt{\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{\sqrt{10}}{5} = \frac{3.162}{5} = .632.$$

2. In the same way find $\sqrt{\frac{1}{7}}$.

Find the value of:

3. $\sqrt{\frac{1}{3}}$.	**5.** $\sqrt{\frac{1}{2}}$.	**7.** $\sqrt{\frac{4}{5}}$.	**9.** $\sqrt{\frac{4}{7}}$.
4. $\sqrt{\frac{2}{5}}$.	**6.** $\sqrt{\frac{1}{10}}$.	**8.** $\sqrt{\frac{1}{6}}$.	**10.** $\sqrt{\frac{5}{8}}$.

Rationalize the denominators:

11. $\sqrt{\frac{7}{8}}$. **13.** $\sqrt{\frac{1}{a}}$. **15.** $\sqrt{\frac{a}{b}}$. **17.** $\sqrt{\frac{1}{ab}}$.

12. $\sqrt{\frac{5}{6}}$. **14.** $\sqrt{\frac{1}{x^3}}$. **16.** $\sqrt{\frac{2}{3a}}$. **18.** $\sqrt{\frac{2b}{3a}}$.

7. ADDITION AND SUBTRACTION OF SURDS

As you have seen in problems 21–23, page 176, the computation in a problem may be greatly simplified when the surds are combined into a single term. Thus,

$$\sqrt{50}+\sqrt{72}+\sqrt{32}=5\sqrt{2}+6\sqrt{2}+4\sqrt{2}=15\sqrt{2}=15\times 1.414.$$

That is, we add the 5, 6, and 4 just as if it were $5a+6a+4a$, and need to look up only the value of 2, instead of the values of $\sqrt{50}$, $\sqrt{72}$, and $\sqrt{32}$.

Simplify:

1. $\sqrt{2}+\sqrt{8}$.

2. $\sqrt{3}+\sqrt{12}$.

3. $\sqrt{18}+\sqrt{27}$.

4. $\sqrt{45}-\sqrt{20}$.

5. $\sqrt{32}-\sqrt{8}$.

6. $\sqrt{50}-\sqrt{18}$.

7. $\sqrt{8}+\sqrt{32}$.

8. $\sqrt{125}-\sqrt{45}$.

9. $\sqrt{252}-\sqrt{28}$.

10. $\sqrt{99}+\sqrt{44}$.

11. $\sqrt{18}-\sqrt{8}+\sqrt{32}$.

12. $\sqrt{75}-\sqrt{3}-\sqrt{12}$.

13. $\sqrt{20}+\sqrt{45}-\sqrt{80}$.

14. $\sqrt{18}+\sqrt{50}-\sqrt{8}$.

15. $\sqrt{40}+\sqrt{90}-\sqrt{10}$.

16. Simplify: $\sqrt{\frac{1}{2}}+3\sqrt{\frac{2}{9}}$.

SOLUTION

$$\sqrt{\tfrac{1}{2}}=\tfrac{1}{2}\sqrt{2};\ 3\sqrt{\tfrac{2}{9}}=3\times\tfrac{1}{3}\sqrt{2}=\sqrt{2}.$$

Hence $$\sqrt{\tfrac{1}{2}}+3\sqrt{\tfrac{2}{9}}=\tfrac{1}{2}\sqrt{2}+\sqrt{2}=\tfrac{3}{2}\sqrt{2}.$$

17. Simplify: $3\sqrt{48} - \sqrt{\frac{1}{3}} - 2\sqrt{\frac{4}{3}}$.

18. Simplify: $\sqrt{\frac{1}{3}} + 4\sqrt{3} - 2\sqrt{\frac{3}{4}}$.

19. Simplify: $2a\sqrt{a^3} + 3\sqrt{a^5}$.

20. Simplify: $3\sqrt{4x^3} + 2x\sqrt{9x} - \sqrt{x^3}$.

8. MULTIPLICATION AND DIVISION OF SURDS

So few problems in elementary mathematics require the multiplication and division of surds that but few exercises are given.

You have learned that $\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9}$. Interchanging the members of the equality,

$$\sqrt{4} \times \sqrt{9} = \sqrt{4 \times 9} = \sqrt{36}.$$

And in general, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

Find:

1. $\sqrt{3} \times \sqrt{5}$.
2. $\sqrt{2} \times \sqrt{5}$.
3. $\sqrt{5} \times \sqrt{7}$.
4. $\sqrt{3} \times \sqrt{11}$.
5. $\sqrt{2} \times \sqrt{6}$.
6. $\sqrt{5} \times \sqrt{15}$.
7. $\sqrt{6} \times \sqrt{12}$.
8. $\sqrt{10} \times \sqrt{30}$.
9. $\sqrt{3} \times \sqrt{14}$.
10. $\sqrt{2} \times \sqrt{8}$.
11. $\sqrt{3} \times \sqrt{27}$.
12. $\sqrt{5} \times \sqrt{125}$.
13. Divide $\sqrt{5}$ by $\sqrt{2}$.

SOLUTION

Expressed as a fraction, this is

$$\frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{2} \times \sqrt{5}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{10}}{2} = \frac{3.162}{2} = 1.581.$$

Using the tables on page 179 *find:*

14. $\sqrt{3} \div \sqrt{2}$.
15. $\sqrt{5} \div \sqrt{3}$.
16. $\sqrt{10} \div \sqrt{2}$.
17. $\sqrt{6} \div \sqrt{3}$.
18. $\sqrt{2} \div \sqrt{7}$.
19. $\sqrt{5} \div \sqrt{10}$.
20. $\sqrt{3} \div \sqrt{7}$.
21. $\sqrt{7} \div \sqrt{10}$.
22. $\sqrt{5} \div \sqrt{6}$.

TABLE OF SQUARES AND SQUARE ROOTS

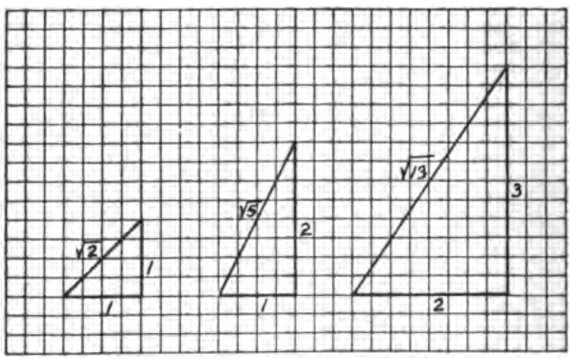
Number	Square	Square Root	Number	Square	Square Root
1	1	1.000000	51	2601	7.141428
2	4	1.414213	52	2704	7.211102
3	9	1.732050	53	2809	7.280109
4	16	2.000000	54	2916	7.348469
5	25	2.236068	55	3025	7.416198
6	36	2.449489	56	3136	7.483314
7	49	2.645751	57	3249	7.549834
8	64	2.828427	58	3364	7.615773
9	81	3.000000	59	3481	7.681145
10	100	3.162277	60	3600	7.745966
11	121	3.316624	61	3721	7.810249
12	144	3.464101	62	3844	7.874007
13	169	3.605551	63	3969	7.937253
14	196	3.741657	64	4096	8.000000
15	225	3.872983	65	4225	8.062257
16	256	4.000000	66	4356	8.124038
17	289	4.123105	67	4489	8.185352
18	324	4.242640	68	4624	8.246211
19	361	4.358898	69	4761	8.306623
20	400	4.472136	70	4900	8.366600
21	441	4.582575	71	5041	8.426149
22	484	4.690415	72	5184	8.485281
23	529	4.795831	73	5329	8.544003
24	576	4.898979	74	5476	8.602325
25	625	5.000000	75	5625	8.660254
26	676	5.099019	76	5776	8.717797
27	729	5.196152	77	5929	8.774964
28	784	5.291502	78	6084	8.831760
29	841	5.385164	79	6241	8.888194
30	900	5.477225	80	6400	8.944271
31	961	5.567764	81	6561	9.000000
32	1024	5.656854	82	6724	9.055385
33	1089	5.744562	83	6889	9.110433
34	1156	5.830951	84	7056	9.165151
35	1225	5.916079	85	7225	9.219544
36	1296	6.000000	86	7396	9.273618
37	1369	6.082762	87	7569	9.327379
38	1444	6.164414	88	7744	9.380831
39	1521	6.244998	89	7921	9.433981
40	1600	6.324555	90	8100	9.486833
41	1681	6.403124	91	8281	9.539392
42	1764	6.480740	92	8464	9.591663
43	1849	6.557438	93	8649	9.643650
44	1936	6.663249	94	8836	9.695359
45	2025	6.708203	95	9025	9.746794
46	2116	6.782330	96	9216	9.797959
47	2209	6.855654	97	9409	9.848857
48	2304	6.928303	98	9604	9.899494
49	2401	7.000000	99	9801	9.949874
50	2500	7.071067	100	10000	10.000000

9. GRAPHS OF QUADRATIC SURDS

Although the exact value of a quadratic surd cannot be computed by arithmetic, by ruler and compasses it may be exactly represented by a line. Since all measure is but an approximation, this method will, in general, not give as close a value as the computed value.

The method depends upon the **Pythagorean Theorem** studied in **Book II.** That is,

The square of the hypotenuse of any right triangle is equal to the sum of the squares of the other two sides.



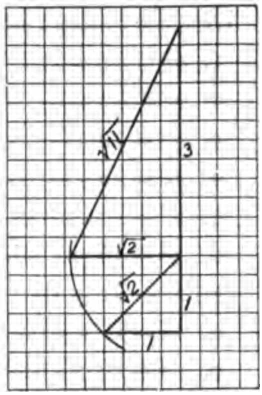
Thus, the figures drawn on squared paper illustrate how to show graphically the length of $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{13}$.

Not all surds can be represented upon a single triangle as in the method shown.

To represent $\sqrt{11}$ two triangles must be made. By studying the following analysis you will see how to make such constructions.

Since $a^2 + b^2 = 11$, we select some square that makes part of 11. There are but three perfect squares less than 11.

These are 1, 4, and 9. Suppose that we let $a^2 = 9$. Then $a = 3$. But this leaves $b^2 = 2$ and $b = \sqrt{2}$. Hence we must first construct $\sqrt{2}$ as shown in the margin. Then a right triangle whose legs are 3 and $\sqrt{2}$ will give a hypotenuse which is $\sqrt{11}$.



With compasses and a ruler construct:

NOTE. — Use but one triangle in exercises 1–6.

1. $\sqrt{5}$.	4. $\sqrt{20}$.	7. $\sqrt{6}$.	10. $\sqrt{38}$.
2. $\sqrt{10}$.	5. $\sqrt{26}$.	8. $\sqrt{18}$.	11. $\sqrt{51}$.
3. $\sqrt{25}$.	6. $\sqrt{41}$.	9. $\sqrt{27}$.	12. $\sqrt{66}$.

10. THE SQUARE ROOT OF TRINOMIALS

You have learned that $(a+b)^2 = a^2 + 2\,ab + b^2$, and that $(a-b)^2 = a^2 - 2\,ab + b^2$. Hence $\sqrt{a^2 + 2\,ab + b^2} = a + b$. What is the square root of $a^2 - 2\,ab + b^2$?

Observe that to be a perfect square the middle term of the trinomial must be twice the product of the square roots of the first and third terms.

Thus, $9\,x^2 - 24\,xy + 16\,y^2$

is a perfect square for

$$24\,xy = 2 \times \sqrt{9\,x^2} \times \sqrt{16\,y^2} = 24\,xy.$$

The square root of

$$9\,x^2 - 24\,xy + 16\,y^2 = \sqrt{9\,x^2} - \sqrt{16\,y^2} = 3\,x - 4\,y.$$

Give the square root of:

1. x^2+6x+9.
2. x^2-4x+4.
3. $y^2-10y+25$.
4. $y^2-18y+81$.
5. $x^2+12x+36$.
6. $y^2+14y+49$.
7. $z^2+8z+16$.
8. $n^2-16n+64$.
9. $x^2+20x+100$.
10. $a^2-22a+121$.
11. $9x^2-24x+16$.
12. $16x^2+56x+49$.
13. $25a^2-60ab+36b^2$.
14. $4x^2+36xy+81y^2$.
15. $4m^2+36mn+81n^2$.
16. $36x^2+24x+4$.
17. $25z^2+90z+81$.
18. $16x^2-56xy+49y^2$.
19. $9a^2-6ab+b^2$.
20. $9a^2-24ab+16b^2$.
21. $x^2+6xy+9y^2$.
22. $n^2+16+8n$.
23. $1-4A+4A^2$.
24. $25t^2+10t+1$.
25. $x^2+6ax+9a^2$.
26. $9v^2-6v+1$.
27. $M^2+4MN+4N^2$.
28. $r^2-10rs+25s^2$.
29. $16H^2-24Ht+9t^2$.
30. $49x^2+4y^2-28xy$.
31. $25W^4-10W^2+1$.
32. $a^4-2a^2b^2+b^4$.
33. $m^2+18ma+81a^2$.
34. $b^2-12bc+36c^2$.
35. $9p^2+42pq+49q^2$.
36. D^4+1+2D^2.

In the following find the term or terms that give a perfect square:

37. $x^2+?+y^2$.
38. $a^2-2ab+?$
39. $?+2mn+n^2$.
40. $4A^2+?+1$.
41. $16-24t+?$
42. $25v^2-?+9w^2$.
43. $W^2+49+?$
44. $36+?-12x$.
45. $a^2-?+100$.
46. $?+1+x^4$.
47. $4P^3+1+?$
48. $r_1^2-12r_1r_2+?$
49. $?-32N+?$
50. $?+42b^2+?$

CHAPTER XII

QUADRATIC EQUATIONS

1. THE FACTORING METHOD

PROBLEMS arise in mathematics that involve the second degree of the unknown. Thus, how can a rectangular plot be laid out to contain 180 sq. ft., and be 3 ft. longer than it is wide?

ANALYSIS OF THE PROBLEM. — Since the dimensions must be known, let $x =$ width, then $x + 3 =$ the length, and $x(x + 3)$ or $x^2 + 3x =$ the area. Then the equation is

$$x^2 + 3x = 180.$$

Or $$x^2 + 3x - 180 = 0.$$

By factoring, we have

$$(x + 15)(x - 12) = 0.$$

If either binomial is equal to zero, the equation is satisfied. For $0 \times (x - 12) = 0$ or $(x + 15) \times 0 = 0$. Hence the equation is satisfied when

$$x + 15 = 0. \qquad (1)$$

Or $$x - 12 = 0. \qquad (2)$$

That is, the solution of either of the simple equations (1) or (2) will satisfy the equation.

Solving (1), $x = -15$.

Solving (2), $x = 12$.

Since a negative length has no meaning, we will take the second solution, $x = 12$. Then $x + 3 = 15$. Hence the dimensions are 12 and 15.

CHECK. — Area $= 15 \times 12$ sq. ft., or 180 sq. ft.

The method of solution used here is the **factoring method.** It depends upon the principle that

If one of the factors of an expression is zero, the whole expression must be zero.

The steps in the factoring method are:

1. *By adding or subtracting, get all terms out of the second member, thus having the first member equal to zero.*

2. *Factor the resulting first member.*

3. *Equate each of the factors to zero, and solve the resulting equations.*

Solve by factoring and check:

1. $x^2 - 6x + 8 = 0.$
2. $x^2 + 4x = 12.$
3. $x^2 - 8x = 9.$
4. $x^2 - 9x = 22.$
5. $x^2 - 15x = 54.$
6. $x^2 + 12x = -20.$
7. $x^2 - x = 6.$
8. $x^2 - 11x = -30.$
9. Find two numbers whose difference is 7 and whose product is 330.

SUGGESTION. — Let x = one, and $x + 7$ the other.

10. Find two consecutive whole numbers whose product is 240.

11. If it takes 184 rods of fence to inclose a rectangular field containing 13 acres, what are the dimensions of the field?

12. By increasing his speed 1 mile per hour, a boy finds that it takes 6 hours less to ride 36 miles. Find his speed.

13. A man being asked what he made on a $56 suit replied, "Just as many per cent of the cost as the suit cost me in dollars." Find the cost and what he made.

14. The area of a triangular sail was to contain 144 sq. ft., and the altitude of the triangle was to be twice the base. Find the base and altitude.

15. Two boys were paid equally to mow a rectangular lawn 60 ft. by 80 ft. How wide a strip entirely around the lawn must the first boy mow in order to mow just half of it?

16. The length of a rectangle is 5 ft. longer than its width. If it contains 300 sq. ft., what are its dimensions?

17. A square box without a lid, and 6 in. deep, is to be made from a square piece of tin by cutting a square from each corner, bending up the sides, and soldering along the edges. If the box is to contain 384 cu. in., find how large a piece of tin is needed.

2. THE FACTORING METHOD NOT SUFFICIENT

You have seen in the problems given that the expressions were often difficult to factor. But there is a greater need for another method. Most problems that lead to quadratic equations are impossible to factor. Those given were so made that factoring was possible. Let us consider the following:

To lay out a rectangular plot 4 rods longer than it is wide, which will contain just $\frac{1}{2}$ acre, will require what dimensions?

SOLUTION

Let x = width, then $x + 4$ = length and $x(x + 4)$ = area. Hence we have the equation

$$x^2 + 4x = 80.$$

Or
$$x^2 + 4x - 80 = 0.$$

By trial we are convinced that no combinations of factors will give a sum of $+4$ and a product of -80. With the factoring method only, the solution is evidently impossible.

3. QUADRATICS SOLVED BY COMPLETING THE SQUARE

If we can make the left-hand member a perfect square without introducing the unknown numbers in the right-hand member, we can reduce the equation to an equation of the first degree by taking the square root of both sides.

We know that any trinomial that is a perfect square is of the form $a^2 + 2ab + b^2$ or of $a^2 - 2ab + b^2$. Thus, in the equation given in the last section, $x^2 + 4x = 80$, $x^2 + 4x$ can be made a perfect square by adding 4. Adding 4 to both members we have

$$x^2 + 4x + 4 = 84.$$

Taking the square root of both members and remembering that 84 has two square roots, $+\sqrt{84}$ and $-\sqrt{84}$, we have

$$x + 2 = +\sqrt{84}. \qquad (1)$$

And $$x + 2 = -\sqrt{84}. \qquad (2)$$

From (1), $$x = -2 + \sqrt{84}.$$

From (2) $$x = -2 - \sqrt{84}.$$

By methods already taught $\sqrt{84} = 9.165$ true to thousandths. (See tables, page 179.)

Then $$x = -2 + 9.165 = 7.165.$$

Or $$x = -2 - 9.165 = -11.165,$$

which has no meaning with reference to length only.

Taking the first value, $x = 7.165$, $x + 4 = 11.165$,

Hence the dimensions are approximately 7.165 rd. and 11.165 rd.

PROOF. — $7.165 \times 11.165 = 79.997225$. The reason that the product is not *exactly* 80 is that the two factors were only approximate values, taken for $\sqrt{84}$. The value taken (9.165) was true only to the nearest thousandth and was slightly too small.

From the formula for a trinomial that is a perfect square, $a^2 + 2ab + b^2$, when $x = a$, the coefficient of $x = 2b$, and the third term b^2 is *the square of half the coefficient of* x.

1. Solve $x^2 - 8x = 12$ by completing the square.

SOLUTION

Adding the square of $\frac{1}{2}$ of -8, or $(-4)^2$ or 16 to both members, we have

$$x^2 - 8x + 16 = 28.$$

Taking the square root of both members,

$$x - 4 = +\sqrt{28}. \quad (1)$$

$$x - 4 = -\sqrt{28}. \quad (2)$$

Hence

$$x = 4 + \sqrt{28}$$

$$x = 4 - \sqrt{28}.$$

In some problems the surd is not evaluated. When the value is needed, the value of $\sqrt{28}$ is found by the tables on page 179, or by other methods that you have had. When possible, use the tables.

The steps in the "completing the square" method:

1. *Reduce the equation to the form* $x^2 + px = q$ *where* x *is the unknown number.*

2. *Add to each member the square of half of the coefficient of the term in the first power of the unknown number.*

3. *Take the square root of both members, observing that the square root of the second member is both* + *and* −.

4. *Solve the two resulting equations.*

5. *Check the solution.*

Solve by completing the square:

1. $x^2 - 6x = -8$.
2. $x^2 - 8x = 9$.
3. $x^2 - 9x = 22$.
4. $x^2 + 14x = -40$.

5. $x^2 + x = 6.$

6. $x^2 + 5x = 14.$

7. $x^2 + 12x + 15 = 0.$

8. $2x^2 + x = 4.$

9. $2x^2 - 5x = 3.$

10. $6y^2 - 13y = -6.$

11. $4a^2 - 11a = 3.$

12. $10n^2 - 29n + 10 = 0.$

13. $3b^2 - 17b = 28.$

14. $2z^2 + 3z = 4.$

15. $5m^2 - m = 7.$

16. $12c^2 - 14c + 3 = 0.$

17. $9x^2 = 6x + 26.$

18. $16a^2 - 96a = 1792.$

19. $N^2 - 7N - 15 = 0.$

20. $N^2 - 2N = 24.$

Solve, finding the value of the roots to thousandths:

21. $x^2 - 3x = 7.$

22. $x^2 - 15x = 60.$

23. $x^2 + 8x = -11.$

24. $x^2 + 6x = 1.$

25. $3x^2 - 4x = 15.$

26. $2x^2 + x = 6.$

27. $2x^2 - 3x + 4 = 0.$

28. $2x^2 - 7x + 6 = 0.$

29. In the center of a rectangular room is a rug 9 ft. by 12 ft. around which is a border of uniform width. If the area of the floor is 208 sq. ft., what is the width of the border?

30. How wide a strip must a boy mow entirely around a rectangular lawn 80 ft. by 120 ft., in order to have it one-fourth mowed?

31. The hypotenuse of a right triangle is 4 in. longer than one leg and 2 in. longer than the other. Find the lengths of all three sides of the triangle.

32. A train running 6 miles per hour slower than usual, owing to a storm, was 1 hr. late in making a run of 252 miles. Find its usual speed.

33. A party arranged a dinner for which they were to pay \$45. Five were absent, and as a result each of those attending had to pay 30 cents more than if all had attended. How many were present?

34. From a board 20 in. square how wide a strip entirely around it must be cut off to leave a square but 64 % as large as the first square?

35. A boy sold his bicycle for \$21, losing as many per cent of the cost as the bicycle cost him in dollars. Find the cost and his loss.

36. The hypotenuse and longer leg of a right triangle are together 18 inches long. The shorter leg is 6 inches long. Find the hypotenuse and longer leg.

37. The area of a rectangle is 192 square inches. The length is 4 inches more than the width. Find the length and width.

38. The area of a triangle is 240 square inches. The altitude is 4 inches less than the base. Find the altitude and base.

39. One side of a rectangle is 8 inches longer than the other. The diagonal is 40 inches. Find the area.

40. When the sides of a square are increased 4 inches, the area is increased 176 square inches. Find the side of the square before it is increased.

41. A rectangular piece of tin was twice as long as it was wide. A 4-inch square was cut from each corner and the sides and ends were turned up so as to make a box containing 1536 cubic inches. Find the dimensions of the piece of tin.

CHAPTER XIII

PROPORTION AND VARIATION

In all mathematics we are dealing with quantitative relationships. In whatever way we state it, the solution of any problem depends upon seeing the relationship of the thing wanted to the thing given. We are commonly dealing with quantities in which a change in one causes a change in the other. Thus, the distance traveled in a given time depends upon the rate of travel. The cost of a quantity of apples depends upon the amount bought, etc. In this chapter we are going to deal with quantities that change together, and to teach the meaning of terms, and methods of stating relationships, that you will need in order to understand references constantly met in mathematics, science, and general reading.

1. THE MEANING OF PROPORTION

In your work in indirect measurement you made use of *ratio*. Thus, in scale drawings you found results from the fact that

The ratio of any two lines in the scale drawings is the same as the ratio of the real lengths which they represent.

And in finding heights and distances by use of similar triangles you used the fact that

The ratio of any two sides of a triangle is equal to the ratio of the corresponding sides of any similar triangle.

To solve the problems, you expressed these equal ratios as an equation. Such an equation, expressing the equality of two ratios, is called a **proportion**. And thus we say that

The corresponding sides of similar triangles are proportional.

This is only another way of expressing the fact given above.

Any equation, then, whose members consist of two ratios is a proportion.

Thus, $\frac{a}{b}=\frac{c}{d}$ is a proportion.

This is read "a over b equals c over d." The four numbers forming the ratios are said to be *in proportion.* The numbers forming one ratio are said to be *proportional to* the numbers forming the other ratio.

2. DEFINITIONS USED IN PROPORTION

In the proportion $\frac{a}{b}=\frac{c}{d}$, a and d are called the **extremes**, and b and c the **means**. If the two means are equal, as in $\frac{a}{x}=\frac{x}{b}$, x is called the **mean proportional** between a and b.

By multiplying both members of the proportion $\frac{a}{b}=\frac{c}{d}$ by bd, we get

$$ad = bc$$

This equation stated in words is one of the well-known laws of proportion, namely,

In any proportional, the product of the extremes equals the product of the means.

The mean proportional between two numbers is the square root of their product. Why?

Find the value of the unknown term in each of the following proportions:

1. $\frac{n}{2}=\frac{15}{10}$.	4. $\frac{7}{9}=\frac{42}{t}$.	7. $\frac{36}{v}=\frac{1}{4}$.
2. $\frac{x}{12}=\frac{1}{8}$.	5. $\frac{3}{10}=\frac{A}{40}$.	8. $\frac{9}{16}=\frac{12}{P}$.
3. $\frac{4}{y}=\frac{16}{21}$.	6. $\frac{24}{35}=\frac{r}{70}$.	9. $\frac{20}{36}=\frac{8}{L}$.

10. A family of three members and a family of four members camp out together. The total cost of provisions is $112, and they wish to divide the cost in proportion to the sizes of the families. How much must each family pay?

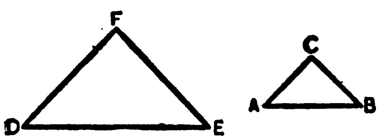
3. FINDING DISTANCES BY PROPORTION

You have learned how to find distances by use of the principle that

The ratio of any two sides of a triangle is equal to the ratio of the corresponding sides of any similar triangle.

It can also be shown that if one side of a triangle bears any constant relation to a similar triangle, all of its sides bear the same relation to the corresponding sides of the similar triangle. This is expressed by saying that

In two similar triangles the corresponding sides are proportional.



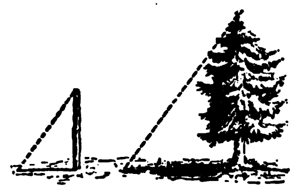
Thus, in the two similar triangles ABC and DEF, this principle states that

$$\frac{AB}{DE}=\frac{AC}{DF}=\frac{CB}{FE}.$$

State the following problems by this principle and solve:

1. In two similar triangles the sides of the larger are 12 in., 14 in., and 20 in., and the shortest side of the smaller is 3 in. Find the other sides of the smaller triangle.

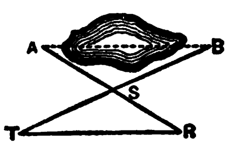
2. The sides of a triangular field are 16 rd., 24 rd., and 30 rd. The smallest side of a similar field is 40 rd. Find the other two sides.



3. A tree casts a shadow 48 ft. long when a vertical rod 6 ft. high casts a shadow 4 ft. long. How high is the tree?

4. A church spire casts a shadow 23 ft. long when a man 5 ft. 10 in. tall, who is passing by, casts a shadow 2 ft. long. Find the height of the spire.

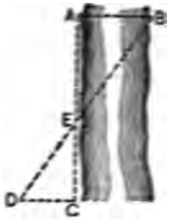
5. The distance from A to B on opposite sides of a lake may be found as follows:



The distances from A to R and from B to T are measured off, making the triangles ASB and TSR similar. If AS is taken 400 yd., and SR 300 yd., and TR is measured and found to be 580 yd., how far is it from A to B?

6. A and B are two points on opposite sides of a hill. Out in the plain at the foot of the hill distances are measured off as in problem 5, so that AS is taken 4050 ft. and SR 1350 ft., and TR is found to be 1600 ft. How far is it between A and B?

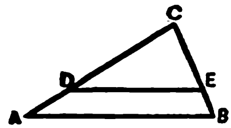
7. To find the distance AB across a stream, measure off a distance AC several yards long, along the bank. Then at C measure off a distance CD at right angles to AC. By sighting across from D to B, locate a stake at a point



E of AC in line with D and B. Get the lengths of AE and EC. Since the triangles are similar, write the proportion by means of which AB may be computed.

If AE is 80 yd., EC 20 yd., and DC 15 yd., what is the width of the stream?

8. It is known in geometry that a line parallel to one side of a triangle divides the other two sides into four proportional parts. That is, if DE is parallel to AB, $\frac{CD}{DA}=\frac{CE}{EB}$.



(1) If $AD=6$ in., $DC=13$ in., and $BE=4$ in., find EC.

(2) If $AD=10$ in., $BE=12$ in., and $EC=20$ in., find DC.

(3) If $DC=9$ in., $BE=7$ in., and $EC=5$ in., find AD.

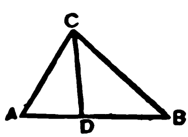
9. In the figure of problem 8, if $AD=8$ in., $DC=5$ in., and $BC=26$ in., find BE and EC.

SUGGESTION.—Let $BE=x$. Then $EC=26-x$.

Hence $$\frac{8}{5}=\frac{x}{26-x}.$$

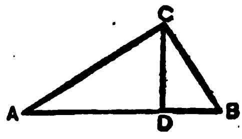
10. In the figure of problem 8, if $AC=40$ ft., $BE=12$ ft., and $EC=15$ ft., find AD and DC.

11. It is known in geometry that in any triangle ABC if the line CD divides the angle at C into two equal parts it divides the opposite side into parts proportional to the other two sides; that is, $\frac{AD}{DB}=\frac{AC}{BC}$. If $AC=8$ in., $BC=10$ in., and $AB=12$ in., find AD and DB.

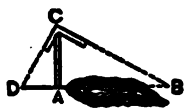


12. In the figure of problem 11, if $AC=40$ yd., $BC=32$ yd., and $AB=60$ yd., find AD and DB.

13. In the triangle ABC, the angle at C is a right angle, and CD is a perpendicular to AB. It is known in geometry that CD is a mean proportional between AD and DB. That is, $\frac{AD}{CD}=\frac{CD}{DB}$. If $AD=20$ in. and $DB=5$ in., find CD.



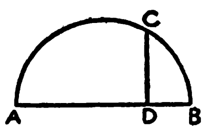
14. A method used several hundred years ago in finding the distance from A to the inaccessible point B was to erect a vertical staff AC, place upon it an instrument resembling a carpenter's square, pointing one blade toward B, and noting the place D on the ground to which the other blade pointed. AC and DA were measured. If $AC=52$ in., and $DA=6$ in., find AB.



SUGGESTION. — This is only an application of the fact stated in problem 13. Why?

15. In the semicircle with the diameter AB, CD, a perpendicular to AB, is a mean proportional between AD and DB.

If $AD=12$ in., and $DB=3$ in., find CD.
If $AD=8$ in., and $CD=6$ in., find DB.
If $AB=20$ in., and $CD=8$ in., find AD and DB.



4. DIRECT VARIATION OR DIRECT PROPORTIONALITY

Any quantity that is constantly changing, as one's age, the population of the country, etc., is called a **variable quantity**, or a **variable.** Any quantity that does not change is called a **constant quantity** or a **constant.** Many of the quantities encountered in every walk of life are variables.

Some variable quantities are so related that the value of one depends upon the value of the other. Thus, the distance d that one can drive in h hours at the rate of 20 miles per hour is found by the equation $d = 20\,h$. That is, d varies with h; as h increases d increases.

The simplest case of related variables is that in which the ratio of the two variables is constant.

Thus, in the above example, when $h = 1$, $d = 20$; when $h = 2$, $d = 40$; when $h = 3$, $d = 60$; etc. That is, the ratio of the number of hours to the number of miles in the corresponding distance traveled is always $\frac{1}{20}$, a *constant.*

Expressed as a formula when x and y are *variables*, and K a constant, this case of variation is represented by the equation or formula

$$\frac{x}{y} = K.$$

In this case x is said to "vary directly" as y. The above equation, by multiplying both members by y, becomes

$$x = Ky.$$

It is evident from the equation for direct variation that

The ratios of any two corresponding values of the variables are equal, or form a proportion.

For, they are each equal to the same constant.

Thus, we say of variables that vary directly that they are *directly proportional.*

Thus, when oranges are 60¢ per dozen, the total cost (C) is "directly proportional" to the number of dozen (D) bought. You know from experience that $\frac{C}{D} = 60$¢, the formula given above for a direct variation.

1. If a train runs 35 miles an hour, write the equation expressing the relation between the variable time t and the distance d.

2. If the price of coffee is 42¢ a pound, the cost of any quantity varies as the number of pounds bought. If $c =$ cost and $w =$ weight in pounds, express the relation between them by an equation.

3. If the price of sugar is 13¢ a pound, express by an equation the relation between the number of pounds bought and the cost.

4. If the rate of interest is 6%, the amount of interest on any sum of money varies as the time. Express by an equation the relation between the interest and the time.

5. The velocity of a body let fall toward the ground varies as the time during which it has fallen from rest, and the velocity at the end of 3 sec. is 96 ft. per second. As the time increases, the velocity increases at the same rate. Write the equation between the velocity and time.

6. The force with which a moving body of any given velocity strikes a stationary body varies as the mass or weight of the moving body. If I strike a nail with a force of 15 lb. by using a hammer weighing $\frac{1}{2}$ lb., with what force would a 2 lb. hammer strike it when swinging with the same speed?

7. If x varies directly as y, and $x = 2$ when $y = 9$, find x when $y = 36$. Write the equation between x and y.

8. If P varies directly as V, and $P = 7$ when $V = 5$, write the equation between P and V. Find V when $P = 35$.

9. If an engine driving a 12-inch pulley gives it 120 revolutions per minute, how many revolutions per minute would it cause a 20-inch pulley to make when running at the same speed?

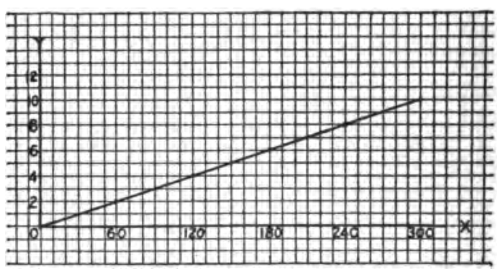
5. A GRAPH OF DIRECT VARIATION

You saw that the equation for direct variation was $x = Ky$ where K was some constant and might be 3, 4, 20, or any definite number. You will notice that this is merely a simple equation of two unknowns, such as you studied in chapter X. You found in that chapter that the graph of such equations is always a straight line. The "price curves" studied in **Book II** are also simple examples of *graphs of direct variation.*

When apples are selling at 30¢ per dozen, the relation of the cost to the number of dozen bought is

$$C = 30\,d.$$

When $d = 1$, $C = 30$; when $d = 5$, $C = 150$. Plotting these points and drawing a straight line through them we have the following graph:



1. Read from the price curve given above the cost of 3 dozen. Of 6 dozen.

2. When eggs sell for 60¢ per dozen, draw the price curve. From the curve, find the cost of 5 dozen. Of 10 dozen.

3. When sugar sells at 12¢ per pound, draw the price curve. From the curve, find the cost of 4 pounds. Of 6 pounds.

4. The circumference of a circle is found by the equation $C = \pi d$ where $\pi = \frac{22}{7}$ approximately. This is an equation of direct variation. That is, "C varies directly as d," or "C and d are directly proportional." Draw a graph of the equation. Find from the graph the value of C when $d = 5$.

5. Write the equation for the area of any triangle whose base is the constant 10, and whose altitude is the *variable* h. Draw a graph of variation.

6. INVERSE VARIATION OR INVERSE PROPORTIONALITY

In our study of variation we have had only the case in which an increase in one variable caused a corresponding increase in the other. Some variables are so related that an *increase* in one causes a corresponding *decrease* in the other. Thus, to drive 90 miles at the rate of 15 miles per hour would require 6 hours. But to *increase* my speed to 30 miles per hour would *decrease* the time required to 3 hours. That is, the time is said to *vary inversely* as the rate of traveling, or the time required is *inversely proportional* to the rate of travel.

In the above example, when $r = 15$, $t = 6$; and when $r = 30$, $t = 3$. That is, their products equalled 90 in each case. And in general,

When two variables vary inversely, their product is a constant quantity.

This is expressed by the equation or formula

$$\boldsymbol{xy = K},$$

where x and y are variables and K a constant.

In the inverse proportion

$$xy = K,$$

for any corresponding value of x and y as x_1 and y_1, x_2 and y_2, etc., we have $x_1y_1 = K$ and $x_2y_2 = K$.

Hence $x_1y_1 = x_2y_2$.

This is only another way of expressing the fact that the product is constant for any values of the variables.

Dividing both members by x_2y_1 we have

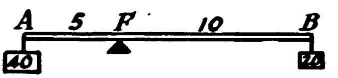
$$\frac{x_1}{x_2} = \frac{y_2}{y_1}.$$

That is,

If one variable is inversely proportional to another, the ratio of any two corresponding values of one is equal to the inverted ratio of any other two corresponding values of the other.

7. PROBLEMS OF THE LEVER

In every simple machine there are two forces involved: the **resistance**, or force to be overcome, and the **effort**, or the force necessary to overcome this resistance. The **lever** is a common example. The lever is a stiff bar of wood or metal that is movable about a fixed point or pivot called the **fulcrum**.

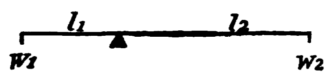


Thus, AB is a lever turning about the pivot or *fulcrum* at F. AF is called the **weight's arm**, and FB the **effort's arm**.

It is a well-known law of the lever that

Weight times weight's arm equals effort times effort's arm.

Using the notation in the margin, this is expressed by the formula



$$w_1l_1 = w_2l_2.$$

Dividing both members by l_1w_2 the equation becomes

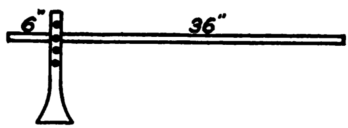
$$\frac{w_1}{w_2} = \frac{l_2}{l_1}, \text{ an inverse ratio.}$$

Hence the law of the lever is often expressed as follows:

In any lever, the resistance and effort are inversely proportional to their distances from the fulcrum.

1. With a "jack," shown in the figure, a man wishes to raise 1200 pounds. How much effort will be required on the long end of the lever?

SOLUTION



By the law as first expressed, we have the equation

$$36x = 6 \times 1200.$$

Solving, $x = 200$.

By the second law, the equation is

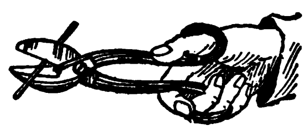
$$\frac{x}{1200} = \frac{6}{36}.$$

Solving, $$x = \frac{1200 \times 6}{36} = 200.$$

Tell which law you prefer to use, and why.

2. What effort is required to raise 800 lb. if the weight (800 lb.) is 5 in. from the fulcrum and the effort is 40 in. from it?

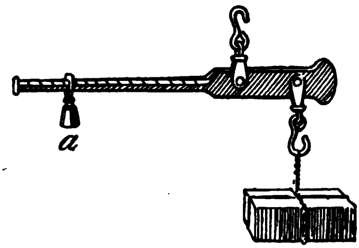
3. What effort is required to raise a stone weighing 600 lb. with a 60-in. crowbar if the fulcrum is placed 6 in. from the stone?



4. The figure shows a pair of heavy shears used in cutting sheet metal or wire. When the wire is 2 in. from the fulcrum (the point where the two parts are joined) and the hand is 8 in. from the fulcrum, what is the cutting force of a squeeze of 15 lb.?

5. Will the wire be more easily cut if moved toward the tip of the blade or toward the fulcrum? Why?

6. For cutting heavy wire would you choose shears with long or short arms, or handles grasped by the hand?



7. This figure shows a pair of steelyards, used in weighing. Suppose that the article to be weighed is 3 in. from the fulcrum and the weight *a* which weighs 1 lb. is moved to a point 24 in. from the fulcrum before it balances the weight of the package. How much does the package weigh?

8. If two boys carry a weight of 150 lb. suspended between them on a 9-ft. pole, find how much each boy is carrying when the weight is 6 ft. from one and 3 ft. from the other.

8. PROBLEMS OF THE WHEEL AND AXLE

The **wheel and axle** consists of a wheel fastened rigidly to a cylinder called the *axle* so that both turn about the same axis. The *effort* is applied to the rim of the wheel and the *weight*, or resistance, is applied to the rim of the axle. It gives the same advantage in lifting heavy weights as the lever does. The law of the wheel and axle is,

The effort times the distance through which it moves equals the weight times the distance through which it moves.

Suppose that the wheel and axle is turned through n revolutions, the effort W' moves $n \times 2\pi r'$ and the weight W through $n \times 2\pi r$. Then we have

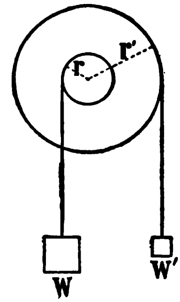
$$2\,Wnr\pi = 2\,W'nr'\pi.$$

Dividing by $2\pi n$,

$$Wr = W'r'.$$

Or, dividing this equation by $W'r$,

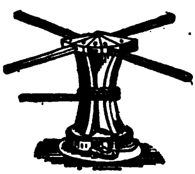
$$\frac{W}{W'} = \frac{r'}{r}, \text{ an inverse proportion.}$$



Stated in words this law is,

The resistance and effort are inversely proportional to the radii of the axle and the wheel, respectively.

1. The crank to a windlass of a well is 18 in. long, and the cylinder upon which the rope is wound has a radius of 3 inches. What force is necessary to lift 48 pounds?



2. A capstan, a form of wheel and axle used in lifting anchors on ships, has an axle 10 in. in diameter and a lever arm or spoke (the radius) 45 in. long. How much effort is required to lift an anchor weighing a ton?

3. A horse walking in a circle 15 ft. in diameter moves a house by means of a capstan 18 in. in diameter. The horse exerts a pull of 1200 lb. What is the resistance of the house?

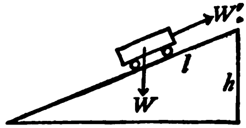
4. Two men working at a capstan walk in a circle 8 ft. in diameter, and each exerts a force of 60 lb. The diameter of the axle is 9 inches. What pull is exerted along the rope?

9. PROBLEMS OF THE INCLINED PLANE

The **inclined plane** is a smooth sloping surface that is used in raising a heavy object through the application of a comparatively small effort. If the effort is applied parallel to the surface of the plane the law of the inclined plane is,

The weight times the height of the plane equals the effort times the length of the plane.

Letting $W' =$ the effort, $W =$ the weight, $h =$ the height, and $l =$ the length of the plane,



$$Wh = W'l.$$

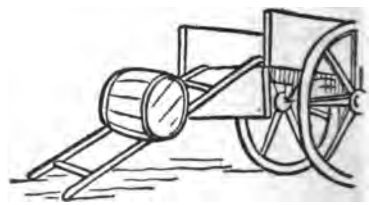
Dividing both members by hW',

$\frac{W}{W'} = \frac{l}{h}$, an inverse proportion.

State the law expressed by this formula.

1. If the length of an inclined plane is 12 ft., the height 3 ft., and the weight of the object lifted 400 lb., what effort is required to move it up the plane?

2. A barrel of material weighing 180 lb. is loaded into a wagon bed 32 in. from the ground by rolling it up a board 10 ft. long with one end resting on the ground and the other on the wagon bed. What effort is required?



3. Two men place a 580 lb. building stone upon a wagon by sliding it up a board 8 ft. long, one end of the board being on the ground and the other resting on the bed of the wagon. The wagon bed is 20 in. from the ground. What effort do they use in loading the stone in addition to that required to overcome the friction?

4. Ice is stored in an ice house by dragging it up an inclined plane. The incline is 200 ft. long and 40 ft. high. What effort is required to pull up a block of ice weighing 250 lb.?

5. A locomotive pulls a train weighing 720 tons up a grade with a rise of $2\frac{1}{2}$ ft. to 100 ft. of grade. How much more pull must the locomotive exert than if the train were on a level road bed?

CHAPTER XIV

RELATIONSHIPS EXPRESSED BY FRACTIONS AND PER CENT

1. EXPRESSIONS OF RELATIONSHIP

YOU have seen throughout mathematics that to give meaning to the quantitative phases of life, relationships are needed. That is, one magnitude must be compared with another.

Two numbers are compared either by *subtraction* or by *division;* that is, their relationship is expressed either as a *difference* or as a *ratio.* Thus, if it is 40 miles to one city and 50 miles to another, we may think of one as being 10 miles nearer than the other, or as but $\frac{4}{5}$ as far away. Or if the population of Los Angeles increased from 319,198 in 1910 to 575,480 in 1920, we can say that it increased 256,282 or about $\frac{4}{5}$, or to be more exact, 80.3 %.

As you have already learned, relationships are shown to the eye by means of *graphs.* But when looking at a bar graph, for example, you see either the difference between the lengths of the bars or the ratio of one bar to the other. Also, when using formulas or equations to express relationships, both differences and ratios are involved. In most of our mathematical thinking, we are more concerned with the ratio relation, expressed either as a *fraction*, a *decimal*, or a *per cent.* This chapter is devoted to the ratio relationships.

2. RELATIONS EXPRESSED BY FRACTIONS

In your early work in fractions you thought of a fraction merely as *one or more of the parts into which some whole had been divided.* Thus, the fraction $\frac{5}{8}$ indicated that some whole had been divided into 8 equal parts and that 5 of these parts made up the fraction of the whole that was considered. Later you used a fraction in division to express the quotient in inexact divisions. Thus, you expressed the quotient of $13 \div 5$ as $2\frac{3}{5}$. That is, you used a fraction *to express division.* You also used a fraction *to express the relation of one magnitude to another.* Thus, if one pair of shoes cost \$7 and another \$12, you expressed the relation by saying that the cheaper pair cost but $\frac{7}{12}$ as much as the other.

Thus we see that

In comparing one number with another, the number compared is measured by the other, taken as the unit of measure. Hence the number compared becomes the dividend (or numerator), and the measure becomes the divisor (or denominator).

1. If you are going on a 24-mile bicycle trip and have gone 18 miles, compare the distance yet to go with the distance gone. With the entire distance.

2. If suits that sold at \$20 five years ago are now selling for \$36, compare the price five years ago with the present price. Compare the increase in price with the price five years ago.

3. If 16 qt. of strawberries make 10 qt. of canned berries, compare the amount of canned berries with the raw. Compare the loss in canning with the amount of raw fruit.

4. When a 12-lb. ham weighs but 9 lb. when boiled, it has lost what part of its weight?

5. When shoes that sold for \$9 are now selling for \$15, the price has increased what part of the original price?

6. If you bought 5 lb. of candy for a party and used but $3\frac{1}{2}$ lb. of it, what part of it did you use?

7. If a man buys 9 bu. of potatoes for the winter and uses all but $1\frac{1}{2}$ bu. of them, he uses what part of what he bought?

8. Helen bought $6\frac{1}{2}$ yd. of goods for a dress and had $\frac{2}{3}$ of a yard left. She used what part of what she bought?

9. Compare the area of a garden plot 80 ft. by 120 ft. with one 90 ft. by 150 ft.

SUGGESTION. — In all such problems it is better to express the relation before performing the multiplication, for like factors may be canceled and thus work be saved. The relation wanted in this problem $= \frac{80 \times 120}{90 \times 150} = \frac{8 \times 4}{3 \times 15} = \frac{32}{45}$.

10. John has the care of a lawn 60 ft. by 80 ft., and Frank has the care of one 80 ft. by 120 ft. Compare John's lawn with Frank's. Compare Frank's with John's.

11. Compare the capacity of a box 10 in. by 16 in. by 24 in. with one 12 in. by 20 in. by 32 in.

12. Compare the capacity of a box 15 in. by 20 in. by 30 in. with one 25 in. square and 15 in. deep.

13. Compare the area of a circle whose radius is 10 in. with one whose radius is 20 in.

14. Compare the area of a circle having a diameter of 20 in. with that of one having a diameter of but 5 in.

15. Compare the size of a cake 8 in. in diameter and 3 in. high with one 12 in. in diameter and 4 in. high.

16. Compare the size of a table glass 2 in. in diameter and 4 in. deep (inside measure) with one 3 in. in diameter and 6 in. deep.

17. Compare the size of an orange 3 in. in diameter with one $4\frac{1}{2}$ in. in diameter, considering them both spherical.

3. INTERPRETING FRACTIONAL RELATIONSHIPS

When a relationship is expressed as a fraction, one should be able to interpret its meaning. The following problems give practice in interpreting such relationships.

1. If you pay 60 ¢ for a remnant of cloth containing $\frac{3}{4}$ of a yard, what is that per yard?

SUGGESTION. — There are a number of ways of analyzing this problem: (*a*) Comparing a whole yard with $\frac{3}{4}$ yd., the relation is $\frac{4}{3}$, hence a yard will cost $\frac{4}{3} \times 60$¢, or 80¢. (*b*) Or, a yard will cost $\frac{1}{3}$ more than $\frac{3}{4}$ yd., hence 60¢ + 20¢, or 80¢. And (*c*) one can reason that if $\frac{3}{4}$ of a yard costs 60¢, $\frac{1}{4}$ of a yard will cost $\frac{1}{3}$ of 60¢, or 20¢, and that 1 yd. will cost 4 × 20¢, or 80¢. (*d*) Using the algebraic equation to state the known relations, let $x =$ the cost of 1 yard, then $\frac{3}{4}x = 60$¢. Solving the equation, $3x = 240$¢, $x = 80$¢. And thus it is seen that there are many ways of solving even a simple problem like the one given.

2. If $\frac{3}{4}$ of a yard of cloth costs 80 ¢, how much will $3\frac{3}{4}$ yards cost? Solve by several methods.

3. I filled my coal bin $\frac{5}{6}$ full at a cost of $145. At the end of the season it is still $\frac{1}{3}$ full. What did the remaining coal cost me?

4. After dishing out 24 dishes of ice cream from a can $\frac{3}{4}$ full, it was still $\frac{3}{8}$ full. How many more dishes of ice cream in the can? Solve without a pencil.

5. After serving 6 people, $\frac{2}{3}$ of a cake is left. How many more can be served? (Do not use a pencil.)

6. Mrs. Brown found that huckleberries lost $\frac{3}{8}$ in canning. How many quarts of canned berries can she get from 16 quarts of raw berries?

7. How many quarts of berries will Mrs. Brown (problem 6) have to buy in order to get 25 quarts of canned berries?

8. A certain kind of small fish lost $\frac{2}{5}$ in dressing. How many pounds of undressed fish will it take to dress 9 pounds?

9. Mr. Jones said: "I used to get $\frac{1}{4}$ more mileage per gallon of gasoline than I now get." If he is now getting 16 miles per gallon, what did he formerly get?

10. A man said: "Our living expenses are $\frac{1}{2}$ more than they were two years ago." If they now average $36 per week, what were they two years ago?

11. When the winter was but half gone a man had used $\frac{3}{4}$ of his fall purchase of coal. At this rate, he will have to buy what part of the amount bought in the fall?

12. It took a man 9 days to harvest $\frac{3}{5}$ of his hay crop. At this rate, how long will it take him to finish?

13. When an article increases in cost from 9¢ to 15¢, the increase is what part of the former cost? What formerly cost $15 will cost how much now?

14. What now costs $25 would formerly cost how much, using the rate of increase of problem 13?

15. The "Fair Price Committee" allowed a merchant to make $\frac{1}{3}$ of the cost on shoes. How much did it allow him to make on a pair of shoes that sold for $12?

16. Suits marked "Special $24" were selling at "$\frac{1}{4}$ off" from a former price. What was the former price?

17. If a merchant is allowed to make only $\frac{1}{4}$ of the selling price of an article, for how much will he have to sell an article costing him $6?

18. If a manufacturer finds that the total cost to produce an article is \$12, for how much must he list it to give him a profit of $\frac{1}{4}$ of the list price?

19. If goods are sold at "$\frac{1}{3}$ off," and still give a profit of $\frac{1}{5}$ of the cost, they were marked to gain what part of the cost?

20. By selling suits at "$\frac{1}{4}$ off" a dealer still made $\frac{1}{5}$ of the selling price. They were marked to make a profit of what part of the selling price? What part of the cost?

21. A profit of $\frac{1}{3}$ of the selling price is a profit of what part of the cost?

22. A profit of $\frac{2}{5}$ of the cost is a profit of what part of the selling price?

23. If an agent makes a gross profit of $\frac{2}{5}$ of the cost of the goods, and his expenses are $\frac{1}{4}$ of his sales, his net profit is what part of the sales?

24. A boy said: "If I sell you this bicycle for \$32, I'll lose $\frac{1}{9}$ of what it cost me." What must it have cost him?

25. In a certain school $\frac{3}{5}$ of the total enrollment were girls. If that is true in a room containing 18 boys, what is the enrollment of that room?

26. An ice-cream can that was $\frac{2}{3}$ full was but $\frac{1}{6}$ full after 36 dishes had been taken out. At this rate, how many more dishes in the can? How many dishes will it hold when full?

27. A merchant received an invoice of goods, costing \$1260, which he marked $\frac{2}{5}$ above cost, but had to sell at "$\frac{1}{4}$ off." How much did he make?

28. If a manufacturer lists his goods $\frac{5}{8}$ above cost but gives a trade discount of $\frac{1}{3}$ of the list price, he makes what part of the cost?

29. An article costs \$90 to manufacture. At what price must it be listed to give a profit of $\frac{1}{4}$ of the selling price after allowing a discount of $\frac{1}{3}$ of the list price?

30. If an automobile runs $\frac{3}{4}$ of a mile in $\frac{4}{5}$ of a minute, what is the rate per hour?

31. A news item says that the population of a certain city increased $\frac{1}{4}$ from 1910 to 1920. If you knew the population in 1910, how could you find the population of 1920? If you knew the population of 1920, how could you find the population of 1910?

32. Knowing that a certain kind of fruit loses $\frac{3}{5}$ in preserving, how can a woman know how much preserved fruit to expect from a given amount of raw fruit? Knowing how much preserves she wants to make, how can she find how much raw fruit to buy? Illustrate your answer by taking a definite amount of fruit both raw and preserved.

33. If a boy buys papers at the rate of 4 for 5¢ and sells them at 2¢ each, he makes how much per 100 papers?

34. If a farmer gathers 330 bushels of potatoes from a field containing $2\frac{3}{4}$ acres, how many bushels can he expect from a field of $5\frac{1}{2}$ acres that are "just as good"? Solve by comparing one field with the other, instead of finding the yield per acre.

35. If $13\frac{1}{2}$ acres of corn produce $630\frac{1}{2}$ bu., what should be expected from 54 acres that are "just as good"? Solve in two ways.

4. APPROXIMATE RELATIONS OF LARGE NUMBERS

To interpret more fully the meaning of many of the large numbers we meet daily in our general reading an *approximate* fractional relation is sufficient. While the relation of 3,000,000 to 8,000,000 is *exactly* $\frac{3}{8}$, the relation of 2,895,378 to 7,953,685 is *approximately* $\frac{3}{8}$, for the first number is nearly

3,000,000 and the second is nearly 8,000,000. These relations are sometimes more easily seen by writing them in fractional form. Thus, to compare 573,840 with 836,375 we may write it $\frac{573840}{836375}$. By drawing a vertical line between any two pairs of digits, thus, $\frac{573|840}{836|375}$, $\frac{57|3840}{83|6375}$, or $\frac{5|73840}{8|36375}$ we see that the relation is *approximately* $\frac{573}{836}$, $\frac{57}{83}$, or that it lies between $\frac{5}{8}$ and $\frac{6}{8}$ or $\frac{3}{4}$. That is, the first number is *nearly* $\frac{3}{4}$ of the second. We are usually concerned with fractions that are made up of one-figured numbers, or two-figured numbers at most.

1. Give the approximate relation of 6,324,500 to 9,687,900 in two-figured numbers. In one-figured numbers.

2. When $\frac{2}{3}$ is given as the *approximate* relation of the numbers given in problem 1, is the relation too large, or too small? Why?

3. In 1920 there was a total expenditure of $1,026,894,700 for good roads, while in 1918 but $610,327,500 was spent. Compare in *approximate* fractions the increase in 1920 over 1918.

SUGGESTION. — Find the increase and compare it with the expenditures of 1918.

4. The population of Chicago increased from 2,185,283 in 1910 to 2,701,212 in 1920, or a total increase of 515,929. This was an increase of *approximately* what part of the population of 1910?

5. If you expressed the relation in problem 4 as $\frac{1}{4}$, is it *nearly* $\frac{1}{4}$, or *more than* $\frac{1}{4}$?

6. The population of Cleveland increased 236,173 from 1910 to 1920. The population in 1910 was 560,663. This was an increase of *about* what fractional part of the population of 1910?

7. If you gave $\frac{2}{5}$ as the relation in problem 6, was that too large or too small? Explain why.

8. The population of Buffalo increased from 423,715 in 1910 to 505,875 in 1920. Give the approximate fractional increase and explain whether your answer is too large or too small.

9. During the fiscal year ending June 30, 1918, we imported 1,052,000,000 pounds of coffee. During the year ending June 30, 1920, we imported 1,500,000,000. This was an increase of about what fractional part of the 1918 importation?

10. During the year 1920 we spent approximately the following sums for things classed as luxuries:

Candy	$1,000,000,000
Cigarettes	800,000,000
Soft drinks	350,000,000
Chewing gum	50,000,000
Total	$2,200,000,000

Give the approximate relation of each to the total.

11. It is estimated that there are about 35,000,000 tons of steel used in the United States yearly. Of this tonnage about 8,500,000 tons are used by the railroads and about 2,500,000 tons are used in the automobile industry. About what part of our total consumption is used in each of these two industries?

12. Before the War, Belgium operated 330,000 flax spindles. In 1920 she had but 103,000 in actual operation. This was about what part of her pre-war record? Explain whether your *approximation* is too large or too small.

13. In a recent year the world's entire production of cotton was 14,126,500 bales, of which all but 2,934,700 bales

was produced in our Southern states. We furnished about what fractional part of the world's supply?

14. The National Safety Council reported that in 1907 one person was killed for every 3,086,319 passengers carried on street cars, and that in 1917 but one person was killed for every 4,393,582 passengers. Does this show that accident hazard is increasing or decreasing?

5. RELATIONS EXPRESSED IN DECIMALS

As you have seen, the exact relation of large numbers is meaningless when expressed as a fraction and to give meaning to it, it must be considered in terms of an *approximate* fraction. This is sufficient only where rough estimates are needed, as in interpreting numbers we meet in general reading. Where more exact relations are needed, they are expressed in decimals or per cents. Thus, to compare the standing of the baseball teams in the American League: Up to July 24, 1920, Cleveland had played 90 games and won 59 of them, while New York had played 94 games and won 61 of them. As fractions, Cleveland's standing was $\frac{59}{90}$ while New York's was $\frac{61}{94}$. From these fractions, it is difficult to compare them, but as decimals, Cleveland's standing was .656 and New York's was .649, from which it is evident that Cleveland was .007 ahead of New York.

1. On July 24, 1920, Chicago's baseball team's record for the season was 90 played and 55 won. Find her decimal standing.

2. Get the baseball statistics from a daily paper at the time you are studying this and work out the standings.

NOTE. — The "Sporting Page" usually labels such standing as "Pct." (per cent) but they are not accurately written as either per cents or decimals. Thus, "Pct. 611" is either .611 or 61.1 %.

3. There are 231 cubic inches in a gallon. Compare a gallon with a cubic foot (1728 cu. in.). A cubic foot with a gallon.

4. There are 2150.42 cubic inches in a standard bushel. Compare a bushel with a cubic foot. A cubic foot with a bushel.

5. Using the result of problem 3, find the capacity in gallons of a tank containing 3450 cubic feet. Which of the two relations found will you use? Why? Explain how you could use either.

6. Using the result of problem 4, find the capacity in bushels of a bin containing 4350 cu. ft. Explain how to use either result found in problem 4. Which result is more easily used?

7. You found in problem 3 that a cubic foot contains 7.4805 gallons if the division is carried to the fourth decimal. This result is approximately what simple "mixed number" expressed in common fractions?

8. We rank first in the production of cotton, producing 11,191,820 bales in a recent year, while British India ranked second, producing 2,949,600 bales. Compare India's crop with ours, carrying the computation to the third decimal.

9. Of the 30,091,500 men engaged in all occupations in a recent year, 734,000 were coal miners. The miners were what decimal part of the whole to the nearest thousandth?

10. Of the total railroad mileage of the world in 1917, the United States had 354,248 miles of the total of 687,123. Give our approximate fractional part of the whole. Give our decimal part to the nearest thousandth.

6. RELATIONS EXPRESSED AS PER CENT

Ratio relations may be expressed in either of three ways, *fractions*, *decimals*, or *per cent*. As you have seen, the fractional way is not convenient or meaningful except when the terms of the fractions are small; and the decimal way, while accurate, involves reading decimals expressed in *thousandths* or even *ten thousandths*. Hence the *per cent* method is much more common. *Per cent*, as you know, merely means *hundredths;* so .35 is written "35%" or "35 per cent"; .358 is written "35.8%" and read "35 and .8 per cent." Likewise .3584 = 35.84%.

In general reading in newspapers and magazines you will observe that the per cent relation is about the only one that is used.

1. The table given below is taken from *The Literary Digest* of July 3, 1920. Check any result that your teacher may select.

POPULATION AND GROWTH OF OUR 18 LARGEST CITIES

City	Population			Increase			
	1920	1910	1900	1910–1920		1900–1910	
				Number	Per Cent	Number	Per Cent
New York City	5,621,151	4,766,883	3,437,202	854,268	17.9	1,329,681	38.7
Chicago	2,701,212	2,185,283	1,698,575	515,929	23.6	486,708	28.7
Philadelphia	1,823,158	1,549,008	1,293,697	274,150	17.7	255,311	19.7
Detroit	993,739	465,766	285,704	527,973	113.4	180,062	63.0
Cleveland	796,836	560,663	381,768	236,173	42.1	178,895	46.9
St. Louis	773,000	687,029	575,238	85,971	12.5	111,791	19.4
Boston	747,923	670,585	560,892	77,338	11.5	109,693	19.6
Baltimore	733,826	558,485	508,957	175,341	34.4	49,528	9.7
Pittsburg	588,193	533,905	451,512	54,288	10.2	82,393	18.2
Los Angeles	575,480	319,198	102,479	256,282	80.3	216,719	211.5
San Francisco	508,410	416,912	342,782	91,498	21.9	74,130	21.6
Buffalo	505,875	423,715	352,387	82,160	19.4	71,328	20.2
Milwaukee	457,147	373,857	285,315	83,290	22.3	88,542	31.0
Washington	437,414	331,069	278,718	106,345	32.1	52,351	18.8
Newark	415,609	347,469	246,070	68,140	19.6	101,399	41.2
Cincinnati	401,158	363,591	325,902	37,567	10.3	37,689	11.6
New Orleans	387,408	339,075	287,104	48,333	14.3	51,971	18.1
Minneapolis	380,498	301,408	202,718	79,090	26.2	98,690	48.7

2. The "statistics" in the margin are taken from *The Cleveland Plain Dealer*, of July 25, 1920. The numbers under "Pct." are not actual per cents. Explain why Cleveland's standing could not be 656%. If she won all she played, it could be only what per cent? Properly express the standings in per cent. Check any standing as your teacher may direct.

3. Bring to class for discussion any reference you find to relations expressed in per cent.

STATISTICS

AMERICAN LEAGUE

Clubs	Plyd.	W.	L.	Pct.
CLEVELAND	90	59	31	656
New York	94	61	33	649
Chicago	90	55	35	611
Washington	84	41	43	488
St. Louis	89	43	46	483
Boston	86	39	47	453
Detroit	85	30	55	353
Philadelphia	92	27	65	293

NATIONAL LEAGUE

Clubs	W.	L.	Pct.	Clubs	W.	L.	Pct.
Brooklyn	53	38	582	New York	42	44	488
Cincinnati	48	37	565	Chicago	44	48	478
Pittsburg	43	40	518	Boston	35	44	443
St. Louis	45	44	506	Philadelphia	36	49	424

AMERICAN ASSOCIATION

Club	W.	L.	Pct.	Club	W.	L.	Pct.
St. Paul	67	27	713	Louisville	46	46	500
Minneapolis	50	45	526	Milwaukee	45	47	489
Indianapolis	48	44	522	Columbus	36	54	400
Toledo	48	47	505	Kansas City	31	61	337

INTERNATIONAL LEAGUE

Club	W.	L.	Pct.	Club	W.	L.	Pct.
Buffalo	60	31	656	Reading	45	47	489
Toronto	59	34	634	Jersey City	34	55	38
Baltimore	58	31	652	Rochester	29	60	326
Akron	56	35	615	Syracuse	20	68	227

7. INCREASES AND DECREASES EXPRESSED IN PER CENT

Almost all references to increases and decreases that you meet in your daily reading are expressed in per cent. To find the per cent of increase or decrease, the actual increase or decrease is first found and that is compared with the thing that increased or that decreased. That was seen in the first table of the preceding section.

1. When shoes that sold for \$8 are now selling for \$18, what is the per cent of increase?

2. Sugar advanced in price from 9¢ per pound to 28¢ per pound within a year. Find the per cent of increase.

3. In the early part of 1920 potatoes advanced in price in some parts of the country from \$2.50 per bushel to \$6 per bushel. Find the per cent of increase.

4. As I am writing this, potatoes are selling in most cities for 10¢ per pound (\$6 per bushel). Find what per cent they have increased or decreased at the time you study this, using the local price of your community.

5. As I write this, sugar is retailing at 28¢ per pound. Find the rate of increase or decrease at the time you study this, using local prices.

6. In a certain industry the cost of labor increased from 70¢ per hour to \$1.75 per hour from 1914 to 1920. Find the per cent of increase.

7. In 1915 we produced 1,011,505,000 bushels of wheat. In 1917 we produced only 659,797,000 bushels. This was a decrease of what per cent?

8. If the price of an article decreases from \$12.75 to \$10.50, what is the per cent of decrease?

9. If the price of a certain make of shoes decreases from \$13.50 per pair to \$9.75 per pair, what is the per cent of decrease?

10. Find references to rates of increase or decrease in your local newspapers and bring to class for discussion and verification.

8. INTERPRETING RELATIONS EXPRESSED IN PER CENT

You constantly see relations expressed in terms of per cent that you must interpret in order to understand fully what you hear and read. Thus I see in the morning paper that wages in a certain corporation have been increased 25 %. For every \$1 received before the increase the employees will receive how much now?

1. I see in a news item that there has been a general increase of 95% in the cost of living. If this is true, what cost $1000 before the increase will cost how much after the increase?

2. One year we produced 659,797,000 bushels of wheat. In the early spring of the next year, the crop reports predicted an increase of 20% for the coming season. That would be an increase of how many bushels?

3. When the yearly cost of food for a family has been $1125, what will the same amount cost when prices advance an average of 15%?

4. The average cost of sugar used by a family of five was only $20.75 in 1914. In 1920 the increase in the price of sugar was 460%. If the average consumption is as great in 1920, what will the sugar cost for a family of five?

5. *The Detroit Free Press*, in a news item, said that the population of Detroit had increased 113.4% from 1910 to 1920. Express as a mixed number in common fractions the approximate relation of the population of 1920 to that of 1910.

6. The population of Detroit in 1920 was 993,739. If she has the same rate of increase from 1920 to 1930 as from 1910 to 1920, what will her population be?

7. The population of Cleveland was 796,836 in 1920, which was an increase of 42.1% since 1910. At this rate of increase, what will her population be in 1930?

8. The population of Philadelphia increased 17.7% from 1910 to 1920. Her population in 1920 was 1,823,158. At the same rate of increase, what will it be in 1930?

9. Take the last five cities named in the table on page 216 and at the rate of increase for the past ten years, see

what the population will be in 1930 and see if it will change their general rank as to relative place in the table.

10. Boston with a population in 1920 of 747,923 showed an increase of 11.5 % since 1910, while Baltimore with a population of 733,926 showed an increase of 34.4 %. With these rates of increase from 1920 to 1930, which city will be larger and how much?

9. INTERPRETING THE RELATION WHEN THE PER CENT GIVEN REFERS TO A NUMBER NOT GIVEN

In our general reading and in our use of mathematics it is much less common to meet a per cent that refers to the number wanted instead of to the number given, as in the problems of the last section, yet such problems do arise. Their analysis is shown here.

1. When the price of an article "including war tax" is 55 ¢, the war tax being 10 % of the price without the tax, how much tax is due the Government in sales, "including tax," amounting to $1254?

ANALYSIS OF THE PROBLEM. — The $1254 includes both the tax and the selling price without a tax. Hence it is 110 % of the price without the tax.

Letting $x =$ the price without the tax, $1.1\,x = \$1254$, hence $x = \$1140$ and the tax $= 10\ \%$ of $1140, or $114.

2. When "movie" tickets are "22 ¢ including 10 % war tax," what is the Government's share of the total receipts from 10,000 tickets?

3. The "Fair Price League" of a certain state allowed dealers in men's suits a maximum profit of 45 % of the cost. Without violating this rule, what is the minimum cost to a dealer of a suit for which he is getting $50?

4. If a dealer is allowed a maximum profit of 35 % of the cost of shoes, shoes selling for $ 12.50 per pair must have cost him a minimum of how much?

5. If a dealer wants to mark an article to include a 10 % war tax and give him a profit of 35 % of the cost on an article costing $ 14.50, for how much must he mark it?

6. In order to allow for a loss of 40 % in dressing, a dealer must buy how much undressed fish to have 150 pounds of dressed fish?

SUGGESTION. — The dressed fish are 60 % of the undressed fish.

7. In order to make 30 % of the selling price, for what must a dealer sell a suit costing him $ 42?

8. A dealer pays $ 7 per ton for ice and sells it at 60 ¢ per 100 lb., but loses 20 % in melting. He is making what per cent of the cost? Of the selling price?

9. A dealer buys potatoes at $5\frac{1}{2}$ ¢ per pound and retails them at 8 ¢ per pound. If there is a loss of 10 % through shrinkage, what per cent of the selling price is he making?

10. If goods costing $ 2800 are sold at a profit of 30 % of the cost, find the profit.

11. If goods costing $ 2800 are sold at a profit of 30 % of the selling price, find the profit.

12. If goods selling for $ 2800 give a profit of 30 % of the selling price, find the profit.

13. If goods selling for $ 2800 give a profit of 30 % of the cost, find the profit.

14. A grocer bought a shipment of peaches at $ 1.80 per basket and sold 90 % of them at $ 2.40 per basket. The remainder spoiled. What per cent of the whole cost did he make?

15. A "green grocer" or dealer in fruits and vegetables reckons on losing 20 % of his purchases. He will have to sell his goods at what per cent above cost to make 30 % of his sales?

16. From what marked price can a merchant deduct $33\frac{1}{3}$ % and still make 20 % of the cost on suits costing him $19.20?

17. From what marked price can a merchant deduct $33\frac{1}{3}$ % and still make 20 % of the selling price on a suit costing him $19.20?

18. If ham, in boiling and slicing, loses 45 % in weight, what is the cost of sliced boiled ham when raw ham costs 48 ¢ per pound?

19. In making white flour, 72 % of the wheat goes to flour. How many bushels of wheat will it take to make 20 barrels of flour? (A bushel of wheat weighs 60 lb. and a barrel of flour weighs 196 lb.)

20. In selling me a suit for $56 a dealer tells me that he is giving me an "out of season" discount of 20 %. From this how can I find the regular price?

10. INTERPRETING PER CENT RELATIONS IN BUYING AND SELLING

Whether engaged in business or not, all our social, civic, and economic thinking requires careful, accurate, quantitative thinking. Franklin Bobbitt says in *The Curriculum* that "one of the fundamental needs of the age upon which we are now entering is acute quantitative thinking in the field of one's own vocation, in the supervision of our many coöperative governmental labors, in our economic thinking with reference to taxation, expenditures, insurance, public utili-

ties, civic improvements, pensions, corporations, and the multitude of other civic and vocational matters." The following problems give practice in seeing and expressing relationships about given data.

1. *Data:* It costs \$76.90 to make a certain article and \$30.76 to market it.

(*a*) The cost to market it is what per cent of the cost to make it?

(*b*) The cost to market it is what per cent of the total cost to make and sell it?

(*c*) It must be sold for how much in order to give a profit of 35 % of the cost to make and sell?

(*d*) It must be sold for how much in order to give a net profit 30 % of the selling price?

(*e*) At what must it be listed in order to give a trade discount of 40 % and leave a profit of 20 % of the cost to make and sell?

(*f*) At what must it be listed in order to allow a discount of 30 % and still have a net profit of 20 % of the selling price?

2. *Data:* The delivered cost of a bill of goods was \$1600.

(*a*) If the buyer sells them at a gross profit of 35 % of the cost, how much does he get for them?

(*b*) If he sells them at a gross profit of 25 % of the selling price, how much does he get for them?

(*c*) If the gross profit is 40 % of the cost and the cost of selling is 25 % of the sales, what is his net profit?

(*d*) In part (*c*), can you find what per cent the net profit is of the cost without using the \$1600?

(*e*) If the gross profit is 30 % of the sales and the cost of selling is 20 % of the sales, what is the net profit?

(*f*) In part (*e*), can you find what per cent the net profit is of the cost without using the \$1600?

3. *Data:* A dealer sold a bill of goods for \$2400.

(*a*) If he made a gross profit of 20 % of the selling price, what was the gross profit?

(*b*) If he made a gross profit of 20 % of the cost, what was the gross profit?

(*c*) If he sold them at a discount of 20 % from the marked price, at what price were they marked?

(*d*) If his gross profit was 40 % of the sales and the cost of selling was 20 % of the sales, what was his net profit?

(*e*) If his gross profit was 50 % of the cost and the cost of selling was 25 % of the sales, what was his net profit?

(*f*) Had he sold the goods at a profit of 25 % of the cost and the cost of selling was 20 % of the sales, did he make or lose and how much? Can you tell what per cent of the cost he made, without using the \$2400?

11. FINDING RELATIONS FROM GIVEN RELATIONS WITHOUT USING DEFINITE NUMBERS

You saw in some of the problems of the last section that relations can be expressed without the use of the definite number given. It is important in our general quantitative thinking that we be able to do this.

1. If wages are increased 40 % and are later reduced 30 %, they are what per cent of the former wage before the increase?

2. If a man's salary is increased 50 % and later decreased 40 %, he is getting how much less than his original salary?

3. If goods sell at a gross profit of 60% of the cost and the cost of selling is 40% of the sales, a merchant is losing what per cent of the cost?

4. If goods are marked to sell at a profit of 40% of the cost and discounted 30%, the loss is what per cent of the cost?

5. A merchant whose overheads amounted to 20% of his sales, marked his goods to sell at 50% above cost, but had to discount them 20%. His loss was what per cent of the cost?

6. After discounting his goods 40%, a merchant still made 20% of the marked price. What per cent above cost were they marked?

7. After discounting his goods 40%, a merchant still made 20% of the selling price. How much above cost were they marked?

8. If goods were listed to give a profit of 40% of the list price, but discounted 20%, they gave a profit of what per cent of the cost? What per cent of the selling price?

9. Goods were sold at a gross profit of 40% of the cost, but the cost of selling was 20% of the sales. The net gain was what per cent of the sales?

10. After discounting goods 25%, a merchant made 20% of the sales. Before discounting them he was making what per cent of the cost?

11. A man marked his goods at an advance of 40% of the cost, but had to discount the marked price 10%. If his total expense of selling the goods is 20% of the sales, what per cent of the cost is he making?

12. Estimating the total cost of doing business to be 20% of the sales, at what per cent above cost must goods be marked to give a net profit of 20% of the cost?

13. After allowing a trade discount of 40% from the list price, a wholesaler is still making 16% of the net selling price. What per cent of the cost is he making? What per cent of the list price is he making?

14. After allowing a discount of 40% of the list price, a wholesaler is still making a gross profit of 20% of the list price. If the cost of doing business is 20% of the selling price, he is making a net profit of what per cent of the list price? Of the selling price? Of the cost?

12. GENERAL USES OF PER CENT

1. A headline in a daily paper said: "Dealers in clothing must limit their profits to 45%." Show that this statement is not definite by showing that if the gain is *of the selling price*, they are allowed a profit of \$22.50 on a suit that is selling for \$50, while if the gain is *of the cost*, they are allowed a profit of but \$15.52 on a suit which they sell at \$50.

2. A man owns and occupies a house that he can sell for \$16,000. His taxes are \$235 per year. The insurance and general upkeep amount to \$85 per year. If he could loan money at 6%, what is it really costing him per year to keep and occupy the house, assuming that the value of the property is not changing?

3. A man had a piece of undeveloped property for which he was offered \$3500. He kept it 2 years, paid out \$120 taxes, and sold it for \$3750. Considering money worth 6%, did it pay him to keep it the two years after he could have sold? Discuss fully, telling why it did or did not.

4. A man bought five \$1000 industrial bonds at 98 paying 6% interest. At the end of 2 years he sold them at 95, paying $\frac{1}{8}$% of the par value both for buying and for selling. Did it pay him, if he could have loaned his money at $5\frac{1}{2}$%?

CHAPTER XV

INTERPRETING STATISTICS

1. THE MEANING AND USE OF STATISTICS

IN all walks of life it is necessary to collect and study certain data that may have any bearing upon the subject considered. Thus, we collect and study such data as school attendance, the cost of living, the wages paid in various industries, the crop productions, the cost of producing a certain article, and data pertaining to thousands of other human interests. When we have a great collection of data bearing upon some question, we call these facts **statistical facts** or **statistics.**

A careful study and comparison of statistics is necessary in the solution of many of our large social, economic, and governmental problems. In **Books I** and **II** you studied graphical methods of picturing data to the eye in order to see the relation of one quantity to another, and you had much practice in studying and interpreting graphs. In this chapter you will learn how to study the **central tendency** of scattered data that make up a table of statistics. Only simple and familiar examples will be considered.

2. A STATISTICAL TABLE OF FACTS

In tabulating data, it is important to arrange the items in an orderly way that will facilitate the ease with which they are studied, and by which comparisons can be made.

The following table shows how the records of two classes

might appear in a teacher's "grade book" in which the names of pupils are recorded alphabetically.

Class A		Class B	
Names	Scores	Names	Scores
Allen, Frank	19	Lewis, James	9
Ames, Walter	7	Lyman, Estella	6
Anderson, Mary	11	Mabey, David	11
Baker, John	8	Miller, Grace	12
Ball, Thomas	16	Newkirk, Frank	7
Bryan, Frank	10	Oliver, George	12
Bush, Ethel	17	Pierce, Helen	8
Davis, William	12	Putman, Mary	18
Dickerson, John	12	Ramsey, Frank	15
Dodd, Walter	10	Russell, Helen	13
Ewing, Ella	9	Sawyer, Lucile	12
Fisher, Henry	15	Stephens, Nellie	16
Foster, Ella	12	Taylor, John	12
Gannon, Frederick	13	Turner, Frank	16
Harrison, Lucile	11	Wilcox, Paul	9
Holmes, George	14	Wright, Arthur	10
Ives, Gertrude	12	Young, Walter	14

From the tables, as they stand, it is very difficult to know which class made the best record, or to tell how the individuals of each class ranked with each other. The comparison is much more simple if the scores are *arrayed* as in the following table.

3. AN ARRAY OR RANK ORDER OF DATA

If the pupils of either class should take their places in line, the one making the highest score taking the head of the line, the one next in order the next, and so on until the one making the lowest score stood at the foot, they are said to be **arrayed** or standing in **rank order**. In such an array,

it is much easier to see how they rank with each, other. This is seen from the following table in which the pupils of the preceding table are *arrayed* in *rank order*.

Class A		Class B	
Names	Scores	Names	Scores
Allen, Frank	19	Putman, Mary . . .	18
Bush, Ethel	17	Stephens, Nellie . . .	16
Ball, Thomas	16	Turner, Frank	16
Fisher, Henry	15	Ramsey, Frank . . .	15
Holmes, George . . .	14	Young, Walter . . .	14
Gannon, Frederick . .	13	Russell, Helen	13
Davis, William . . .	12	Miller, Grace	12
Dickerson, John . . .	12	Oliver, George	12
Foster, Ella	12	Sawyer, Lucile . . .	12
Ives, Gertrude . . .	12	Taylor, John	12
Anderson, Mary . . .	11	Mabey, David . . .	11
Harrison, Lucile . . .	11	Wright, Arthur . . .	10
Dodd, Walter	10	Lewis, James	9
Bryan, Frank	10	Wilcox, Paul	9
Ewing, Ella	9	Pierce, Helen	8
Baker, John	8	Newkirk, Frank . . .	7
Ames, Walter	7	Lyman, Estella . . .	6

4. COMPARING DATA BY MEANS OF A SINGLE NUMBER

When data are scattered, as in the tables of the last two sections, we need single numbers which tell us the **central tendency** of the group or the particular value about which the values center. Thus, in the *array* of the last section, if we count down from the top, or up from the bottom to the middle score, or to the ninth score in this case, we see that the middle score of each class is 12. That is, there were 8 above, and 8 below, the pupil that made this score. This is called the **median**. That is,

If any group of data is arrayed, the middle one is known as the **median.** *If there is an even number of items, the median exists halfway between the two middle items.*

The *median* is easily found from an array and gives a fairly good measure of the *central tendency* of the group. In the example just given, however, it does not show any difference between Classes A and B as a whole, for the median of each is 12.

By finding a total of all scores made by each class, we find that Class A made a total of 208, while Class B made but 200. By dividing the total score of each by the number of pupils, we get the **arithmetical average** of each class. A's *average* is $12\frac{4}{17}$, while B's is only $11\frac{13}{17}$. That is,

The quotient obtained by dividing the sum of all the items by the number of items is known as the **arithmetical average.**

1. From 20 exercises in addition written upon the blackboard, each consisting of 5 addends of 2 figures each, let the class see how many they can copy and add correctly in just 8 minutes. Array the scores, find the median and the average score.

2. Do the same with exercises in subtraction, using numbers of 5 digits.

3. Do the same with multiplication, using a 2-figured multiplier and a 3-figured multiplicand.

4. Do the same with the scores of a spelling test of 50 words dictated by your teacher.

5. Beginning with exercise 1, page 157 of this text, see how many exercises you can solve in just 10 minutes. Find the median and average of the class.

6. Do the same with the exercise on page 187.

5. ARRANGING DATA IN FREQUENCY TABLES

When a very large number of items, many of which are alike, make up the data, it is impractical to array them. In such cases they are tabulated as to the frequency of each item. For example, 1300 pupils of a city might make scores ranging from 2 to 10 on a test in addition. For the purpose of studying the central tendency, these would first be tabulated in a **frequency table.**

A FREQUENCY TABLE OF 1300 SCORES IN ADDITION

No. of Pupils	Scores	No. of Pupils	Scores
15	10	225	5
70	9	150	4
120	8	100	3
200	7	20	2
400	6		

When thus arranged in a *frequency table* we have a new single number that fairly well represents the central tendency. Thus we see that 6 was the *most common* score, for more made that score than any other.

In a frequency table, the commonest number is known as the **mode** (taken from the French "*la mode,*" meaning "*the fashion*"). In the above table, 6 is the *modal score*, and requires no calculation.

By adding 15, 70, 120, and 200 we have 405, so we see that the *median score* is also 6, for the middle one would fall in the group of 400 that made a score of 6.

To find the average we must multiply each item by its frequency before adding. Thus, from the above table we find

the sum of $15 \times 10 + 70 \times 9 + 120 \times 8 + 200 \times 7 + 400 \times 6 + 225 \times 5 + 150 \times 4 + 100 \times 3 + 20 \times 2$. And thus we find the *average* to be 6.38. (Check this.)

You have now had three single numbers that fairly well represent the *central tendency* of the entire group. They are: *the arithmetical average; the median;* and *the mode.* The three numbers usually are not alike, and their reliability, or the closeness with which each represents the central tendency of the group, depends upon the distribution of the data.

1. In seeking the "central tendency," as to size, of a new variety of tomatoes, a gardener weighed 181 taken at random from his garden one morning and tabulated the result as follows:

Wt. to nearest oz.	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Frequency of no. of tomatoes	1	3	5	12	20	25	35	22	20	14	10	9	3	1

(*a*) Find the average size. (*b*) Find the median size.
(*c*) What is the modal size? (*d*) How nearly do the results agree?

6. ADVANTAGES AND DISADVANTAGES OF THE THREE MEASURES OF CENTRAL TENDENCY

The arithmetical average: Where there are no great extremes existing among the data, the *average* perhaps gives the best notion of the central tendency. But in case of extreme scores the result is misleading. For example, in measuring the ability of a class, a very high or a very low score would have too much emphasis upon the average. Thus, if a class of 10 made the following scores, 1, 6, 6, 8, 8, 8, 8, 9, 10, and 10, the average score is 7.4. But omitting the very low score of 1, the average of the remaining 9 is 8.11. A greater

extreme will make this still more noticeable. Suppose that 10 men subscribe \$1450 to a certain charity, one man giving \$1000 and the other 9 giving but \$50 each. The average is \$145 each, which in no way represents the central tendency, for the *average* is nearly three times as much as was given by any but one of the group. Both the mode and median here is \$50, which gives a far better notion of the character of the donations.

The median: The median has the advantage of eliminating the undue influence of extreme cases. It shows the number from which there is an equal number higher and lower. In simple arrays like the ones shown in this chapter it requires only counting from either end of the array until the middle item is reached. In a study of more complex data, more difficult calculation is required. Such calculations are omitted here as having no value to the average citizen not working with such statistics.

The mode: The mode eliminates the extremes. An unusually high or low item has no effect on the mode. On the other hand, there is a large number of frequency tables in which it cannot be applied with satisaction. Thus, there may be several groups that have nearly the same frequency, but none of them really outstanding. When there is a single outstanding group, the mode is a very satisfactory measure of the central tendency.

7. CLASS LIMITS AND CLASS INTERVALS

In arranging data covering a large number of items, it would be extremely difficult to represent or tabulate each item, and it would give too large a group of figures to study. Thus, in the exercise of section 5 the "weight to the nearest ounce" was taken of each tomato. By so doing, we had

only 14 different weights to consider. Whereas, if we had taken the *exact* weight of each of the 181 tomatoes, we might have had nearly as many different weights. A more common way is to group the numbers between **class limits.** Thus, instead of the "nearest" ounce, we might have grouped all weighing from 1 oz. to 2 oz.; from 2 oz. to 3 oz.; etc. Or, suppose that we are making a study of the "central tendency" as to weight, of boys of a certain age, it would be well to group them within certain limits. For, out of 1000 or 10,000 boys of a given age, weights may range all the way from 60 lb. to 140 lb. Even to group within a **class interval** of 1 lb. (that is, to put all weighing from 60 lb. to 61 lb. into one class, from 61 lb. to 62 lb. into another, and so on) would require 81 classes. In studying so large a group, a class interval of 5 lb. or 10 lb. would be more convenient.

Thus, the following table shows the method of grouping the weight of boys of a certain age in a *class interval* of 10 lb.

Wts. in Lbs.	No. of Boys	Wts. in Lbs.	No. of Boys
55–65	5	95–105	350
65–75	35	105–115	180
75–85	102	115–125	80
85–95	200	125–135	12

1. Give the class limits in the above table.

2. In what class interval does the mode fall?

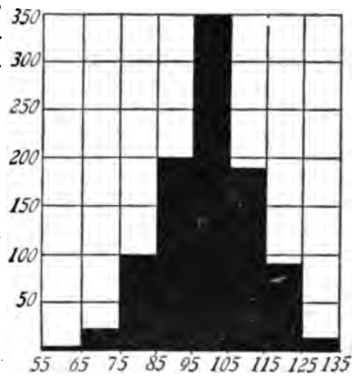
3. In what class interval does the median fall?

4. Study the table and tell why you would have expected both the mode and median to fall in the same class interval. Make up a table in which they would not be the same.

8. FREQUENCY POLYGONS AND SURFACES

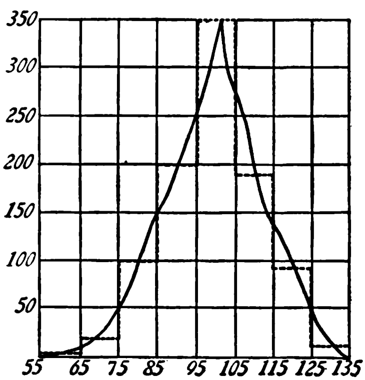
The preceding table may be represented by bar graphs, as shown in the graph in the margin.

The *area* covered by the bars of the graph is called a **frequency polygon** or a **frequency surface.**



1. If the class interval had been 5 lb. instead of 10 lb., in what way should you expect it to change the general shape of the area of the polygon here represented? In what way should you expect a class interval of 1 lb. to change it?

It is very probable that if the *exact* weight of each individual boy could have been represented in the graph, the only change would be that the broken line inclosing the frequency surface would have seemed more nearly a smooth curve, being broken into so many more parts. That is, it would more nearly be the heavy curve of the figure in the margin.



Thus it is seen that the central tendency of a large number of items may be shown as well by grouping into class intervals as by considering each individual item.

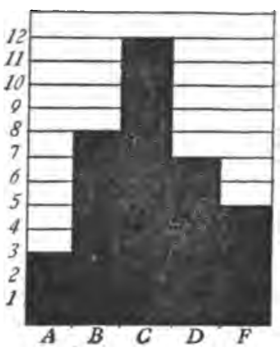
9. NORMAL DISTRIBUTION

In grouping any human trait or ability in a frequency table, it will be seen that there is a gradual rise and fall in numbers from the very lowest to the very highest. Thus, in a spelling test of 100 words, a few will get a very low score, a few a very high score, and the largest number will be neither extreme.

1. Weigh 20 boys of the same age, arrange the weights in a frequency table with a class limit of 5 lb. Construct a frequency polygon, and see how it compares in general shape with the one given.

2. Get the height of twenty boys and arrange a frequency table with a class interval of 2 in. Draw a frequency polygon.

3. Let one member of your class give you a spelling test of 100 words. Make a frequency polygon of the scores made. Does it in any way resemble the graph given in the text?

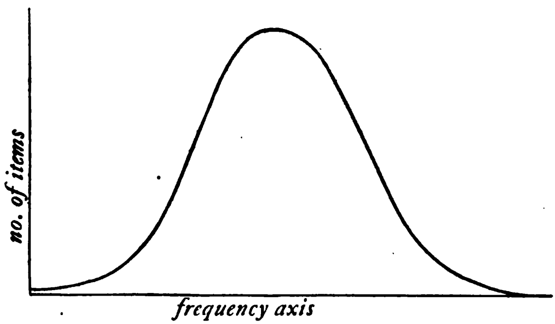


4. Get the results of tests and make graphs and see if they resemble the following, which is a graph of the grades of a class of 35.

3 pupils made *A*.
8 pupils made *B*.
12 pupils made *C*.
7 pupils made *D*.
5 pupils made *F* (failed).

If we measure the length of ears of corn from the same field, the height of the stalks, the production of certain

crops, etc., we see that the data are distributed very much as in human traits; that is, the rise and fall on either side of the *mode* is fairly regular, and the curve would approximate the ones you found above and the one found in section 8. Such a distribution is called a **normal distribution**, and the curve that results when the class intervals are small enough to give a smooth curve as shown in section 8 is called the **curve of distribution.**



The curve above is the ideal *frequency curve of distribution* toward which all frequency areas approach, whether measuring human traits or those of nature.

5. The following table shows the distribution of intelligence quotients made by 2360 primary children given in a certain city by buildings. Make a "frequency polygon" of the scores of any building, as your teacher may direct, and see how nearly they agree in general shape. Do they *approximate* the "normal curve" shown above when a smooth line connects the ends of the various bars?

6. Make a "frequency polygon" of the total scores in the table and see how nearly it agrees in general shape with that of the scores of some building which you have found.

INTELLIGENCE QUOTIENTS	BUILDING NUMBER														TOTAL
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
Below 56 . . .	1	1	0	1	1	1	1	1	0	1	0	0	2	0	10
56– 65 . . .	5	0	5	5	6	2	1	0	0	6	0	0	3	0	33
66– 75 . . .	18	5	13	17	8	10	7	4	14	3	2	1	2	1	105
76– 85 . . .	28	20	20	38	20	28	17	13	13	18	16	2	8	2	243
86– 95 . . .	62	42	45	68	36	50	39	22	36	28	38	16	36	8	526
96–105 . . .	49	34	37	72	48	41	60	29	55	50	50	25	66	22	638
106–115 . . .	29	18	36	48	33	48	48	36	50	38	45	19	51	23	522
116–125 . . .	3	9	10	11	12	14	11	7	20	24	20	13	28	17	199
126–135 . . .	0	0	4	2	1	4	1	2	9	8	8	2	16	9	66
136–145 . . .	0	1	0	0	1	0	1	2	1	4	1	0	3	2	16
Above 145 . .	0	0	0	0	0	0	0	0	0	2	0	0	0	0	2
Total . .	195	130	170	262	166	198	186	116	198	182	180	78	215	84	2,360
Median . .	93	94	95	95	97	97	100	101	101	102	102	103	104	109	99

7. At sight, give the class interval within which the modal score of each building falls.

8. The median score of each building may be found as follows:

BUILDING NO. 1. — The median is the score of the 98th pupil. Why? There are 52 pupils in the four groups that made below 86. Hence we must count 46 of the 62 pupils that fall within the group "86–95," in order to get the score of the 98th pupil. Assuming the marks evenly distributed, there will be an increase of 1 score for each group of 6 pupils (nearly). Since $46 \div 6 = 7\frac{2}{3}$, counting 46 of this group will take us into the 8th group of 6, or add 8 points to the 85, (the score of the 52d pupil), making the score of the 98th pupil 93, the *median* score.

In the same way, find the median score of each building, and thus check the medians given in the table.

NOTE. — There are other methods of computing the median, too complex to discuss here.

INDEX

ANSWERS: BOOK III

Pages 6–7

1. $2s + 2t$.
2. $A = f + i$.
3. $n + m$.
4. $t - c$.
5. $d + t - n$.
6. $2a + 3b$.
7. $\frac{1}{2}(x - 2y)$.
8. $a^2 - b^3$.
9. $3ab$.
10. $\frac{1}{2}(2a + 5b)$.
11. $a^2 + b^2$.
12. $(a + b)^2$.
13. $a^2 - b^2$.
14. $(a - b)^2$.

Pages 10–13

1. 1.224^-.
2. $14\frac{6}{11}$.
3. 8 lb.; 9600 lb.
4. 16; 64; 144; 256; 1600; 57,600; 1,440, 000.
5. $21\frac{1}{9}$; $17\frac{7}{9}$; $3\frac{1}{3}$; $26\frac{2}{3}$.
6. 508.7^-.
7. 160.8^+.
8. 30.7^+.
9. 4; 10; 20.
10. 76; $195\frac{5}{16}$.
11. 480 lb.
12. 1692^+ gal.

Pages 14–15

4. 105 ft.
5. 8 rd.
6. 37.5 ft.
7. 34.13^- in.; 23.89^- in.
8. 4 yd.
9. 5.1^+ ft.
10. $3\frac{1}{8}$ in.
11. 7.4^+ ft.
12. 5 ft.
13. 16 rd.

Pages 22–23

1. 3.
2. 3.
3. $\frac{3}{8}$.
4. 3.
5. 7.
6. 4.
7. 2.
8. 3.
9. 3.
10. 2.
11. 5.
12. 1.
13. 2.
14. 5.
15. 8.
16. 1.
17. 12.
18. 10.
19. 180.
20. 6.
21. 18.
22. 18.
23. $14\frac{1}{7}$.
24. 24.
25. 90.
26. 45.
27. 105.
28. 5.

Pages 27–30

19. \$23.
20. \$95.
21. 8.
22. \$10.
23. \$56.
24. \$4800.
25. 36 mi.
26. 40¢; 80¢.
27. 8 boys.
28. 20 and 25.
29. 21.
30. 16 in. and 24 in.
31. 60.
32. 60 and 105.
33. 78 and 21.
34. 36 yr., 96 yr.
35. 112 yd., 224 yd.
36. 32, 96.
37. \$240, \$120.
38. A, 1200; B, 1500; C, 900.
39. 48.
40. 400 A.
41. 12 and 48.
42. 5.
43. A, \$1000; B, \$500; C, \$1600.
44. 6 da.
45. 6 hr.
46. 7.
47. 7.
48. \$1350.
49. A, \$77; B, \$42.
50. 70 mi.
51. 54°, 54°, 72°.
52. 8.
53. 27 and 21.
54. 16.
55. 24 in.

Page 31

1. $b = \frac{A}{h}$.
2. $h = \frac{2A}{b}$.
3. $d = \frac{C}{\pi}$.
4. $r = \frac{C}{2\pi}$.
5. $r = \frac{i}{pt}$.
6. $h = \frac{V}{bw}$.
7. $h = \frac{V}{\pi r^2}$.
8. $h = \frac{3V}{b}$.
9. $h = \frac{2V}{\pi r^2}$.
10. $h = \frac{A}{b_1 + b_2}$.
11. $S = \sqrt{A}$.
12. $r = \sqrt{\frac{A}{\pi h}}$.
13. $r = \sqrt{\frac{V}{\pi h}}$.
14. $r = \sqrt{\frac{3V}{\pi h}}$.
15. $r = \sqrt{\frac{S}{4\pi}}$.
16. $r = \sqrt[3]{\frac{3V}{4\pi}}$.
17. $a = \sqrt{h^2 - b^2}$.
18. $a = \sqrt{C + b^2}$.

Page 39

1. 1329.3^+.
2. 791.3^-.
3. 197.8^-.
4. 59.6^-.
5. 1814.0^- (1813.99).
6. 1034.7^+.
7. 1448.3^+.
8. 3389.3^+.
9. 68.0^- (67.96^-).
10. 613.9^-.

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6. 150°.
7. 30°, 60°.
8. 40°, 50°.
9. 70°, 110°.
10. $33\frac{1}{3}$°, $56\frac{2}{3}$°.

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6. acute.
7. right.
8. obtuse.

Pages 63–64

6. $60°$.
7. $70°$.
8. $60°$.
10. $30°$, $60°$, $90°$.
11. $18°$, $72°$.
12. $80°$.
13. $35°$, $55°$.
14. $58\frac{1}{3}°$, $43\frac{1}{3}°$, $78\frac{1}{3}°$.
15. $20°$, $40°$, $120°$.
16. $18°$, $72°$.
17. $50°$.

Pages 71–73

1. $+3$.
2. $+1$.
3. $+6$.
4. $+1$.
5. -2.
6. $+4$.
7. $+3$.
8. $+8$.
39. $+80$.
40. -19.
41. $+123$.
42. $+166$.
43. -19.
44. -125.
45. -106.

Page 76

32. $+35$.
33. -22.
34. $+59$.
35. -51.
36. -54.
37. $+33$.
38. $+42$.

Pages 81–82

1. $+5a$.
2. $-6b$.
3. $+2x$.
4. $-3y$.
5. $-6c$.
6. $+10e$.
7. $-13m$.
8. $+4n$.
9. 0.
10. $+6z$.
11. $-4y$.
12. 0.
13. $-3r$.
14. $-10x$.
15. $+9u$.
16. $-4t$.
17. $+3x$.
18. $+6m$.
19. $-3m$.
20. $-7n$.
21. $+12s$.
22. $-13s$.
23. $+21x$.
24. $+3y$.
25. $-20r$.
26. $+23t$.
27. $+16r$.
28. $-24t$.
29. $+18s$.
30. $+8m$.
31. $-5t$.
32. $+50x$.
33. $10x-y$.
34. $-11r-4s$.
35. $6m+n$.
36. $11x-12y$.
37. $3x+13y$.
38. $-2t+3n$.
39. $13r-6s$.
40. $16r+5s$.
41. $-9x-2y$.
42. $-5m-3n$.
43. $5x-5y$.
44. $3t+5s$.
45. $6m+8n$.
46. $9r-9s$.
47. $9r+2s$.
48. $6x-y$.
49. $9x-7y$.
50. $-4m+3s$.
51. $6n-8t$.
52. $5x+6y$.

Pages 83–84

1. 7.
2. 6.
3. 10.
4. 6.
5. 28.
6. $2x+1$.
7. $3y-4$.
8. $4x-5y$.
9. $2x-2y$.
10. $2s-5t$.
11. $4a+11b$.
12. $9r+5s$.
13. $5x+5y$.
14. $16y+5x$.
15. $x=11$.
16. $x=5$.
17. $x=5$.
18. $x=4$.
19. 13 and 7.
20. 12 and 5.
21. $56\frac{2}{3}°$, $76\frac{2}{3}°$, $46\frac{2}{3}°$.
22. $a=105°$, $b=75°$.
23. $55°$.
24. $50°$, $100°$, $30°$.

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1. $-6ab$.
2. $+6ab$.
3. $+6bc$.
4. $-15ab$.
5. $+6abc$.
6. $-8ab^2c$.
7. $-6a^2bc$.
8. $+8x^2y$.
9. $+12abc$.
10. $+24xyz$.
11. $+12abc$.
12. $+12a^2b^2c^2$.
13. $-6abcd^2$.

Pages 90–91

1. $3x-6y$.
2. $2ab+6ac$.
3. $6x^2-4xy$.
4. $21ab-28a$.
5. $16c^2-40c$.
6. $12a-14b$.
7. $6a^2-15ab$.
8. $3ac-21bc$.
9. $-10a+15c$.
10. $8a-4c$.
11. $-6ac+2ab$.
12. $-3x^2-2xy$.
13. $-abc+3b^2c$.
14. $-20c-24cd$.
15. $6ac-15a^2b$.
16. $2x+13b+2a$.
17. $8x-14y$.
18. $2a+17b$.
19. $3t+33s$.
20. $-3ab+6ac$.
21. $ac-11bc$.
22. $-xy+4xz+yz$.

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1. 9.
2. 5.
3. 6.
4. 2.
5. 5.
6. 7.
7. 4.
8. 3.
9. 7.
10. 5.
11. 8 and 12.
12. $40°$, $50°$.

Page 93

1. $2x^2+7x+6$.
2. $2x^2-5x-3$.
3. $2n^2+n-6$.
4. x^2-x-12.
5. $6n^2+5n-6$.
6. $2x^2+3xy-2y^2$.
7. $6a^2-7ab-3b^2$.
8. $6a^2-13ab+6b^2$.
9. $4n^2+4nm-15m^2$.
10. $10n^2+nm-2m^2$.
11. $8x^2+14xy-15y^2$.
12. $2m^2-16m+30$.
13. $a^2+2a-35$.
14. $6a^2-20a-16$.
15. $2b^2+7bc+3c^2$.
16. $2a^2+5ab-12b^2$.
17. $3x^2+14xy+15y^2$.
18. $3x^2-19xy-14y^2$.
19. $2m^2-3mn-9n^2$.
20. $6t^2+11tu-10u^2$.
21. $2r^2-7rs-30s^2$.
22. $r^2-6rt-16t^2$.
23. $4d^2+23de-6e^2$.
24. $2a^2+ab-6ac-3bc$.
25. $6ac-2bc+12ad-4bd$.
26. $2ac-6bc-5ad+15bd$.

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1. $2a^2+11a+15$.
2. $2a^2-13a+15$.
3. $6a^2+7a-3$.
4. $6x^2-13x-5$.
5. $2x^2+11x-21$.
6. $3c^2+11c-4$.
7. $6c^2+13c+6$.
8. $6c^2-13c+6$.
9. $12c^2-11c-2$.
10. $4b^2+16b+15$.
11. $2a^2+7ab+3b^2$.
12. $6a^2+7ab-3b^2$.
13. $10x^2-xy-2y^2$.
14. $15x^2-13xy+2y^2$.
15. $4a^2+13ac+3c^2$.
16. $10a^2+21ac-10c^2$.
17. $10a^2-21ac-10c^2$.
18. $8x^2-6xy+y^2$.
19. $12y^2-23ay+5a^2$.
20. $8x^2-34xy+21y^2$.

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1. a^6.
2. x^4.
3. s^2.
4. n^4.
5. $2b$.
6. $2c^2$.
7. a^2b.
8. a^4.
9. $2a^3$.
10. $5x^2y$.
11. $4a^2$.
12. $2a^4$.
13. $-3a^4$.
14. $-5a^4b$.
15. $-4x^4y$.
16. $-2x^2y^3$.
17. $-4x^2y$.
18. $-4ac$.
19. $6bc$.
20. $-4a^2$.
21. $-6c$.
22. $7x^2$.
23. $-7b$.
24. $-3m^2$.
25. $-19s$.
26. $-9n^2$.
27. $3bc$.
28. $-7yz$.
29. $2x$.
30. $-2mn$.
31. $-16a$.
32. $-9x^4$.

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1. $27b-6c$.
2. $-6x+8y$.
3. $c-2b+3d$.
4. $-y^2+3xy-2$.
5. $-4a^3+3a^2x-2ax^2$.
6. $xy-1+4x^2$.
7. $-x^7+5x-3$.
8. $a^2-2ab+3b^2$.
9. $-x+1$.
10. $3x^4-2x^2$.
11. $a-b+c$.
12. $xy^3+2x^4y^2$.
13. $2x^2y-3x^3$.
14. $a^2+\dfrac{3b}{a}-6b^2$.
15. $-2n^2+3m$.
16. $2n-3m^2n^5$.

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1. $a+2$.
2. $N-5$.
3. $x-3$.
4. $P+4$.
5. $t-2$.
6. $y-5$.
7. $H+7$; 26 rem.
8. $b-4$.
9. $m+13$; 90 rem.
10. $x+1$.
11. $3x-2$.
12. $3k+1$.
13. $2V+5$.
14. $2A-3$.
15. $5R+2$.
16. $2p-1$.
17. $1+3a$.
18. $4-5D$.
19. $a+b$.
20. $x-2y$.
21. $W+5g$.
22. $m+n$.
23. $P-4Q$.
24. $x+7$.
25. $a+1$.
26. $y+11$; -8 rem.
27. $6E^2+1$.
28. $r^2-3r+6\frac{1}{2}$; $-16\frac{1}{2}$ rem.
29. B^2+B+1.
30. y^2-yz+z^2; $-2z^2$ rem.
31. W^2-V^2.
32. H^3-t^3.
33. $12R-1$.

Page 114

1. 7.
2. -4.
3. $2\frac{5}{6}$.
4. $5\frac{1}{3}$.
5. 2.
6. $\frac{1}{2}$.
7. $3\frac{1}{2}$.
8. $\frac{1}{4}$.
9. $15\frac{1}{7}$.
10. 5.
11. $\frac{1}{2}$.
12. 7.
13. 2.
14. $\frac{1}{2}$.
15. -17.
16. $1\frac{1}{31}$.
17. -5.
18. 8.
19. 7.
20. -1.

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3. $(10x+7)(x+1)$.
4. $(3x+1)(x+3)$.
5. $(5a+2)(a+2)$.
6. $(2b+5)(b+2)$.
7. $(a+1)(3a+7)$.
8. $(2x+3)(4x+5)$.
9. $(3a-5)(a-2)$.
10. $(10c-7)(c-1)$.
11. $(3x-1)(x-3)$.
12. $(5a-2)(a-2)$.
13. $(2b-3)(4b-5)$.
14. $(2y-5)(y-2)$.
15. $(2c-1)(c-5)$.
16. $(3t-2)(t-2)$.
17. $(4r-9)(r+3)$.
18. $(3x-2)(2x+7)$.
19. $(3a+5)(a-1)$.
20. $(3x+4)(x-1)$.
21. $(3b-2)(b+1)$.
22. $(4y-5)(y+1)$.
23. $(3c-5)(c+1)$.
24. $(2x-7)(x+4)$.

Page 122

1. $a(a^2+3a+1)$.
2. $(2x+3y)(2x+3y)$.
3. $n(m+3)(m+3)$.
4. $(a+2)(a+6)$.
5. $a(t-2)(t-6)$.
6. $(y+6)(y-2)$.
7. $a(n-6)(n+2)$.
8. $a(a+3)(a+5)$.
9. $t(t+5)(t-3)$.
10. $a(x-2y)(x+2y)$.
11. $r(r-5)(r+3)$.
12. $a(n+6)(n+7)$.
13. $(a+7)(a-6)$.
14. $(2a+1)(a+1)$.
15. $x(2y+1)(y+2)$.
16. $(t+13)(t-2)$.
17. $(3a+1)(2a+3)$.
18. $(x-8)(x-8)$.
19. $a(y+4)(y+4)$.
20. $(x+7y)(x-5y)$.

Page 123

1. $-\frac{a}{b}$.
2. $-\frac{x}{y}$.
3. $\frac{x}{y}$.
4. $-\frac{2a}{3b^2}$.
5. $-\frac{6xy}{5z}$.
6. $\frac{7n}{12m^2}$.
7. $-\frac{9x}{11y}$.
8. $\frac{at}{8m}$.
9. $-\frac{3}{5xy}$.
10. $-\frac{a+b}{a-b}$.
11. $-\frac{x(y+z)}{y(x+z)}$.
12. $-\frac{a(c+b)}{b(a-c)}$.

Page 124

1. $\frac{5}{6}$.
2. $\frac{3}{4}$.
3. $\frac{5}{6}$.
4. $\frac{3}{4}$.
5. $\frac{3}{5}$.
6. $\frac{3}{8}$.
7. $\frac{7}{9}$.
8. $\frac{18}{25}$.
9. $\frac{3}{10}$.
10. $\frac{1}{12}$.
11. $\frac{b}{c}$.
12. $\frac{1}{2x}$.
13. $\frac{2m}{3n}$.
14. $\frac{3p}{4q^3}$.
15. $\frac{2c}{3ad}$.
16. $\frac{xz}{2}$.
17. $\frac{w}{4v}$.
18. $\frac{t}{3r}$.
19. $\frac{2M}{9v^3}$.
20. $\frac{4}{R}$.
21. $\frac{1}{2R}$.
22. $\frac{a^2-b^2}{a^2+3ab+b^2}$.
23. $\frac{x}{n+m}$.
24. $\frac{A(A-B)}{A+B}$.
25. $\frac{R-1}{2y}$.
26. $\frac{t+5}{t+2}$.
27. $\frac{1+p}{1-p}$.
28. $\frac{2n+1}{2n-1}$.
29. $\frac{v-w}{v+w}$.

Page 125

1. $1\frac{2}{3}$.
2. $1\frac{5}{7}$.
3. $3\frac{1}{4}$.
4. $1\frac{7}{8}$.
5. $p+1+\frac{2}{p-1}$.
6. $2x+1+\frac{2}{2x-1}$.
7. $x-y+\frac{2y^2}{x+y}$.
8. $x^2-xy+y^2-\frac{2y^3}{x+y}$.
9. $n-3-\frac{5}{n-2}$.
10. $1-x+\frac{2x^2-2x^3}{1+x-x^2}$.
11. $x^2-2x+4+\frac{8}{x+2}$.
12. $y+1-\frac{y-6}{y^2-y+2}$.

Pages 125–126

1. $\frac{11a}{8}$.
2. $\frac{3a}{2b}$.
3. $\frac{abd+ac}{cd}$.
4. $\frac{15x+2ax}{10a}$.
5. $\frac{27x-14}{35}$.
6. $\frac{z+y}{xyz}$.
7. $\frac{a^2+b^2}{abc}$.
8. $\frac{x^2+y^2}{x^2-y^2}$.
9. $\frac{2a^2+2b^2}{a^2-b^2}$.

Page 126

1. $\frac{ax}{my}$.
2. $\frac{4acm}{5bd}$.
3. $\frac{5a}{n}$.
4. $\frac{bm^2}{ad}$.
5. $\frac{abx}{2}$.
6. 2.
7. $\frac{a^3c}{bd^2}$.
8. $\frac{x^5}{ab}$.
9. $\frac{z}{x}$.
10. $\frac{a^3}{c}$.
11. $\frac{3ac}{(x+y)^2}$.
12. $a+b$.

Pages 126–127

1. $\frac{2a^2}{3m}$.
2. $\frac{y}{2m}$.
3. $\frac{2a}{3m}$.
4. $\frac{8n}{9m}$.
5. $\frac{3x}{my}$.
6. $\frac{5x}{7a}$.
7. $\frac{25xz}{6y}$.
8. $\frac{35a^2m}{4n^2}$.
9. $\frac{y(a+b)}{x(a-b)}$.
10. $\frac{a}{a-1}$.
11. $\frac{2b}{a-b}$.
12. $\frac{6a+2}{a^2(2-a)}$.

Pages 127–128

3. 1.
4. 6.
5. 1.
6. 9.
7. 1.
8. −1.
9. 6.
10. 15.
11. $2\frac{2}{3}$.
12. 1.
13. $4\frac{4}{5}$ da.
14. $1\frac{5}{7}$ hr.
15. 6 hr., 12 hr.
16. 9 min.
17. 3 ft., 9 ft.

Pages 130–131

1. 250 yd.
3. 60 ft.; 80 ft.; 100 ft.
5. 128 yd.
6. About 365 yd.
7. About 127 ft.
8. Nearly 80 yd.
9. About 88 yd.
10. Nearly 185 yd.
11. About 152 yd.

Pages 135–136

1. 1 in. = 20 yd.
2. 15 yd.
3. 1 in. = 50 ft.
4. 600 ft.
7. 1 in. = 40 ft.
8. 800 ft.
9. 600 ft.

Pages 137–138

1. About 115 ft.
3. About 35 ft.
4. Nearly 34°.
5. About 725 ft.
6. About 56 ft.; about $18\frac{1}{2}$ ft.
7. About 67 ft.
8. 3500 ft.
9. About 3400 ft.
10. Nearly 90 ft.

Pages 139–142

2. 80 ft.
3. $66\frac{2}{3}$ ft.
5. $133\frac{1}{3}$ ft.
8. $6\frac{1}{4}$ ft.
11. 320 yd.
13. 83 ft.
16. 30 ft.

Pages 147–150

2. 128.56 yd.
3. 257.12 yd.
4. 114.72; 222.93; 321.677.
6. Nearly 52°.
8. 77.032 ft.
10. 195 ft., nearly.
11. 59 ft., nearly.
12. 176.34 yd.
13. 906.3 ft.; 422.6 ft.
14. 549.5 ft.
15. 668.5 yd.
16. 329.1225 ft.
18. 4502 ft.
19. 62.78 ft.
20. 72.8 ft.

Page 151

1. $A = \frac{1}{2}\,ac \sin B.$
 $A = \frac{1}{2}\,ab \sin C.$
2. 14,776.8 sq. ft.
4. 25,500 sq. ft.

Pages 153–155

2. 135.57 ft.
3. 192.8 yd.
4. 1423.23 yd.
5. 93.98 ft.;
 18.03 ft.

Pages 157–158

1. $x = 2;\ y = 5.$
2. $x = 2;\ y = 3.$
3. $x = 1;\ y = -4.$
4. $x = -1;\ y = 6.$
5. $x = -2;$
 $y = -3.$
6. $x = 5;\ y = 1.$
7. $x = 4;\ y = 2.$
8. $x = -2;$
 $y = -3.$
9. $x = 5;\ y = 7.$
10. $x = 5;\ y = -5.$
11. $x = 4;\ y = 3.$
12. $a = 3;\ b = 7.$
13. $x = 12;\ y = 11.$
14. $x = 1;\ y = \frac{1}{2}.$
15. $x = 11;\ y = -3.$
16. $x = 3,\ y = 4.$
17. $x = -2;\ y = 5.$
18. $x = 6;\ y = 1.$
19. $x = 5;\ y = 3.$
20. $x = 5;\ y = 1.$

Pages 158–159

1. $4\frac{1}{2}$ ft.
2. 20 rd. by 30 rd.
3. 20 ft. by 40 ft.
4. $4\frac{1}{2}$ ft., $5\frac{1}{2}$ ft.
5. 6 ft., 8 ft.
6. 17 boys, 25 girls.
7. 468 Jr. H. S.;
 312 Sr. H. S.
8. 8 lb. at 30¢;
 12 lb. at 24¢.
9. 20 dimes, 30 nickels.
10. 37° and 53°.

Pages 160–161

1. $x = 3;\ y = 2.$
2. $x = 8;\ y = 4.$
3. $x = 3;\ y = 1.$
4. $x = 17;\ y = 13.$
5. $x = 6;\ y = 2.$
6. $x = 1;\ y = 5.$
7. $x = -3;$
 $y = -7.$
8. $x = 9;\ y = 2.$
9. $a = -11;\ b = 7.$
10. $a = 8;\ b = 1.$
11. $r = 2;\ s = 1.$
12. $r = 8;\ s = -2.$

Pages 161–162

1. 15 and 6.
2. 4 and 9.
3. 24 and 96.
4. 8 and 56.
5. $\frac{14}{21}.$
6. $85\frac{5}{7}$ ft. and
 $114\frac{2}{7}$ ft.
7. 10 dimes,
 40 nickels.
8. $53\frac{1}{3}°$, $126\frac{2}{3}°$.
9. 30°, 60°.
10. 36°, 72°, 72°.
11. 4 gal. milk,
 16 gal. cream.
12. 83. 13. 27.

Page 163

1. $x = 1;\ y = 1.$
2. $x = -2;\ y = 3.$
3. $x = \frac{1}{3};\ y = \frac{1}{2}.$
4. $x = 3;\ y = 4.$
5. $a = 3;\ b = 2.$
6. $a = 2;\ b = 5.$
7. $r = 6;\ s = 4.$
8. $x = \frac{25}{134};$
 $y = -\frac{10}{33}.$

Page 164

1. 4 hr., 6 hr.
2. 3 da., 5 da.
3. 8 hr., 12 hr.
4. 16 min.,
 20 min.
5. $10\frac{25}{32}$ min.,
 $17\frac{2}{11}$ min.
6. 56 hr., 105 hr.
7. 30 da., 24 da.

Pages 173–174

1. $\pm 5.$ 3. $\pm 6.$
2. $\pm 9.$ 4. $\pm 4.$
5. $\pm 8.$
6. $\pm 11.$
7. $\pm 15.$
8. $\pm 10.$
9. $\pm 2.$
10. $\pm 2.$
11. $\pm 5.$
12. $\pm 9.$
13. 12 ft.
14. 7 ft.
15. 15 ft. by 30 ft.

Page 175

1. $5\sqrt{3}.$
2. $3\sqrt{3}.$
3. $5\sqrt{2}.$
4. $2\sqrt{10}.$
5. $5\sqrt{5}.$
6. $6\sqrt{5}.$
7. $5\sqrt{7}.$
8. $10\sqrt{3}.$
9. $a\sqrt{a}.$
10. $x^2\sqrt{x}.$
11. $b^3\sqrt{b}.$
12. $c^4\sqrt{c}.$
13. $2a\sqrt{2a}.$
14. $4x^2\sqrt{2x}.$
15. $2y\sqrt{7y}.$
16. $2ab^2\sqrt{3ab}.$

Pages 175–176

1. 8.660.
2. 8.944.
3. 9.486.
4. 17.320.
5. 26.832.
6. 14.694.
7. 13.230.
8. 12.726.
9. 15.652.
10. 18.972.
11. 15.810.
12. 21.168.
13. 15.588.
14. 16.
15. 34.640.
16. 86.600.
17. 56.560.
18. 69.280.
19. 52.960.
20. 97.960.
21. 18.382.
22. 13.856.
23. 17.888.

Pages 176–177

2. .378. 6. .316.
3. .577. 7. .894.
4. .632. 8. .408.
5. .707. 9. .756.
10. .791.
11. $\frac{1}{4}\sqrt{14}.$
12. $\frac{1}{6}\sqrt{30}.$
13. $\frac{1}{a}\sqrt{a}.$